

MINIMAL MODEL THEORY FOR RELATIVELY TRIVIAL LOG CANONICAL PAIRS

KENTA HASHIZUME

ABSTRACT. We study relative log canonical pairs with relatively trivial log canonical divisors. We fix such a pair $(X, \Delta)/Z$ as above, and we establish the minimal model theory for (X, Δ) assuming the minimal model theory for all Kawamata log terminal pairs whose dimension is equal to the dimension of Z . We also show the finite generation of log canonical rings for log canonical 5-folds which are not of log general type.

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1. INTRODUCTION

Throughout this paper we work over \mathbb{C} , the complex number field.

In the minimal model theory for higher-dimensional algebraic varieties, one of the most important problems is the existence of a good minimal model or a Mori fiber space for log pairs. In this paper we only deal with the case when the boundary divisor is a \mathbb{Q} -divisor.

Conjecture 1.1. *Let (X, Δ) be a projective log canonical pair such that Δ is a \mathbb{Q} -divisor. If $K_X + \Delta$ is pseudo-effective then (X, Δ) has a*

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good minimal model, and if $K_X + \Delta$ is not pseudo-effective then (X, Δ) has a Mori fiber space.

Conjecture 1.1 for log canonical threefolds is proved by Kawamata, Kollár, Matsuki, Mori, Shokurov and others. Conjecture 1.1 for Kawamata log terminal pairs with big boundary divisors is also proved by Birkar, Cascini, Hacon and McKernan [BCHM]. But Conjecture 1.1 is still open when the dimension is greater than three.

One of interesting situations to consider Conjecture 1.1 is when (X, Δ) is a relative log canonical pair whose log canonical divisor is relatively trivial. The situation is a special case of lc-trivial fibration, which is expected to play a crucial role in inductive arguments. For example, Ambro's canonical bundle formula for Kawamata log terminal pairs, which is proved by Ambro [A], gives an inductive argument. On any klt-trivial fibration $(X, \Delta) \rightarrow Z$, which is a special case of lc-trivial fibration, Conjecture 1.1 for (X, Δ) can be reduced to Conjecture 1.1 for a Kawamata log terminal pair on Z by the canonical bundle formula. Ambro's canonical bundle formula is expected to hold for log canonical pairs in full generality but it is only partially solved (cf. [F5], [FG1] and [FG3]).

In this paper we establish an inductive argument for log canonical pairs in the above situation. The following is the main result of this paper.

Theorem 1.2. *Let $\pi : X \rightarrow Z$ be a projective surjective morphism of normal projective varieties and let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor. Suppose that $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$. Set $d = \dim Z$ and assume Conjecture 1.1 for all d -dimensional Kawamata log terminal pairs.*

Then Conjecture 1.1 holds for the pair (X, Δ) .

The following theorem follows from Theorem 1.2.

Theorem 1.3 (=Theorem 6.2). *Let (X, Δ) be a projective log canonical pair such that Δ is a \mathbb{Q} -divisor and the log Kodaira dimension $\kappa(X, K_X + \Delta)$ is nonnegative. Let F be the general fiber of the Iitaka fibration and (F, Δ_F) be the restriction of (X, Δ) to F . Suppose that (F, Δ_F) has a good minimal model.*

If (X, Δ) is Kawamata log terminal or $\kappa(X, K_X + \Delta) \leq 4$, then (X, Δ) has a good minimal model.

As a corollary of Theorem 1.3, we have

Corollary 1.4. *Let (X, Δ) be a projective log canonical pair such that $\dim X = 5$ and Δ is a \mathbb{Q} -divisor. If (X, Δ) is not of log general type,*

then the log canonical ring $\mathcal{R}(X, K_X + \Delta)$ is a finitely generated \mathbb{C} -algebra.

We recall some previous results related to Theorem 1.2 or Corollary 1.4. In [GL], Gongyo and Lehmann established an inductive argument for \mathbb{Q} -factorial Kawamata log terminal pairs (X, Δ) with a contraction $f : X \rightarrow Z$ such that $\nu((K_X + \Delta)|_F) = 0$, where $\nu(\cdot)$ is the numerical dimension and F is the general fiber of f . More precisely, in the situation, they proved existence of a Kawamata log terminal pair $(Z', \Delta_{Z'})$ such that Z' is birational to Z and (X, Δ) has a good minimal model if and only if $(Z', \Delta_{Z'})$ has a good minimal model. On the other hand, Birkar and Hu [BH] proved existence of a good minimal model for log canonical pairs (X, Δ) when $K_X + \Delta$ is the pullback of a big divisor on a normal variety whose augmented base locus does not contain the image of any lc center of (X, Δ) . In particular they proved existence of a good minimal model for all log canonical pairs (X, Δ) when $K_X + \Delta$ is big and its augmented base locus does not contain any lc centers of (X, Δ) . On the other hand, Lai [L] proved Theorem 1.3 in the case when X has at most terminal singularities and $\Delta = 0$. Related to Corollary 1.4, Fujino [F2] proved the finite generation of log canonical rings for all log canonical fourfolds. In the klt case, Birkar, Cascini, Hacon and McKernan [BCHM] proved the finite generation of log canonical rings in all dimensions.

In Theorem 1.2, the case when $X = Z$ implies equivalence of Conjecture 1.1 for Kawamata log terminal pairs and Conjecture 1.1 for log canonical pairs (see also [FG4]). If (X, Δ) is Kawamata log terminal, then Theorem 1.2 follows from Ambro's canonical bundle formula for Kawamata log terminal pairs (cf. Proposition 4.3). But since we do not assume that (X, Δ) is Kawamata log terminal, we can not use the canonical bundle formula directly. We also note that we do not have any assumptions about lc centers of (X, Δ) in Theorem 1.2. We hope that Theorem 1.2 will play an important role in inductive arguments for the minimal model program.

We outline the proof of Theorem 1.2. We prove Theorem 1.2 by the induction on the dimension of Z . Note that Conjecture 1.1 for all d -dimensional Kawamata log terminal pairs implies Conjecture 1.1 for all d' -dimensional Kawamata log terminal pairs such that $d' \leq d$ (cf. Lemma 4.1), and we can assume that $K_X + \Delta$ is pseudo-effective. First, we take a special dlt blow-up and replace (X, Δ) with the dlt model. Then we can decompose $\Delta = \Delta' + \Delta''$. Next we construct a morphism $\pi' : (X', \Delta_{X'}) \rightarrow Z'$ which is birational to $\pi : (X, \Delta) \rightarrow Z$ and replace π with π' . Then we may assume that $\Delta'' \sim_{\mathbb{Q}, Z} 0$ and

$K_X + \Delta$ is the limit of movable divisors. Moreover we can apply Ambro's canonical bundle formula to (X, Δ') . For details, see Section 4. With those preparations, we divide the theorem into three cases. But the goal is similar in each case. We pick a sufficiently small rational numbers $u_1 > 0$ and $u_2 > 0$, and construct a modification $\pi' : (X', \Delta_{X'}) \rightarrow Z'$ of $\pi : (X, \Delta) \rightarrow Z$ such that $(X, \Delta) \dashrightarrow (X', \Delta_{X'})$ is a finitely many steps of the $(K_X + \Delta - u_i \Delta')$ -MMP for $i = 1, 2$. Then there is a contraction $X' \rightarrow W$ such that $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}, W} 0$ and $\dim W < \dim Z$. By the induction hypothesis $(X', \Delta_{X'})$ has a good minimal model. Then we see that (X, Δ_X) has a good minimal model by construction. For details, see Section 5.

The contents of this paper are the following. In Section 2 we collect some definitions and notations. In section 3 we introduce the definition of D -MMP, where D is \mathbb{R} -Cartier and not necessarily log canonical divisor, and prove some related result. In Section 4 we reformulate Theorem 1.2 and prepare some results to prove Theorem 1.2. In Section 5 we prove Theorem 1.2. In Section 6, we prove Corollary 1.4.

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2. PRELIMINARIES

In this section we collect some notations and definitions. We will freely use the definitions and notations in [KM] and [BCHM] except the definition of models (see Definition 2.5). Here we write down only some important notations and definitions, including the notations not written in [KM] or [BCHM].

2.1 (Divisors). Let $\pi : X \rightarrow Z$ be a projective morphism of normal varieties and let $D = \sum d_i D_i$ be a \mathbb{Q} -divisor. Then D is a *boundary \mathbb{Q} -divisor* if $0 \leq d_i \leq 1$ for any i . The *round down* of D , denoted by $\lfloor D \rfloor$, is $\sum \lfloor d_i \rfloor D_i$ where $\lfloor d_i \rfloor$ is the largest integer which is not greater than d_i . Suppose that D is \mathbb{Q} -Cartier. Then D is called a *log canonical divisor* if D is the sum of the canonical divisor K_X and a boundary \mathbb{Q} -divisor. D is *trivial over Z* , denoted by $D \sim_{\mathbb{Q}, Z} 0$, if D is \mathbb{Q} -linearly equivalent to the pullback of a \mathbb{Q} -Cartier \mathbb{Q} -divisor on Z . D is *anti-ample over Z* if $-D$ is ample over Z . In this paper we mean the same definition by saying that D is *trivial* (resp. *anti-ample*) *with respect to π* . D is *semi-ample over Z* if D is a $\mathbb{Q}_{\geq 0}$ -linear combination of semi-ample Cartier divisors over Z , or equivalently, there exists a morphism

$f : X \rightarrow Y$ to a variety Y over Z such that D is \mathbb{Q} -linearly equivalent to the pullback of an ample \mathbb{Q} -divisor over Z .

For any \mathbb{Q} -divisor on X , we define the sheaf of \mathcal{O}_Z -algebra

$$\mathcal{R}(X/Z, D) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor mD \rfloor).$$

We simply denote $\mathcal{R}(X, D)$ when Z is a point. If D is a log canonical divisor, then $\mathcal{R}(X/Z, D)$ is nothing but the log canonical ring.

Similarly we can define boundary divisors, log canonical divisors, triviality over Z , semi-ampleness over Z , and so on for \mathbb{R} -divisors.

Let $X \dashrightarrow Y$ be a birational map of normal projective varieties and let D be an \mathbb{R} -divisor on X . Unless otherwise stated, we mean the birational transform of D on Y by denoting D_Y or $(D)_Y$.

2.2 (Singularities of pairs). Let X be a normal variety and Δ be an effective \mathbb{R} -divisor such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) . Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta) E_i$$

where E_i are prime divisors on Y and $a(E_i, X, \Delta)$ is a real number for any i . Then we call $a(E_i, X, \Delta)$ the *discrepancy* of E_i with respect to (X, Δ) . The pair (X, Δ) is called *Kawamata log terminal* (*klt*, for short) if $a(E_i, X, \Delta) > -1$ for any log resolution f of (X, Δ) and any E_i on Y . (X, Δ) is called *log canonical* (*lc*, for short) if $a(E_i, X, \Delta) \geq -1$ for any log resolution f of (X, Δ) and any E_i on Y . (X, Δ) is called *divisorial log terminal* (*dlt*, for short) if Δ is a boundary \mathbb{R} -divisor and there exists a log resolution $f : Y \rightarrow X$ of (X, Δ) such that $a(E, X, \Delta) > -1$ for any f -exceptional prime divisor E on Y .

We recall the construction of dlt models. The following theorem is proved by Hacon.

Theorem 2.3 (Dlt blow-ups, cf. [F3, Theorem 10.4], [KK, Theorem 3.1]). *Let X be a normal quasi-projective variety of dimension n and let Δ be an \mathbb{R} -divisor such that (X, Δ) is log canonical. Then there exists a projective birational morphism $f : Y \rightarrow X$ from a normal quasi-projective variety Y such that*

- (i) Y is \mathbb{Q} -factorial, and
- (ii) if we set

$$\Gamma = f_*^{-1} \Delta + \sum_{E: f\text{-exceptional}} E,$$

then (Y, Γ) is dlt and $K_Y + \Gamma = f^(K_X + \Delta)$.*

2.4 (Dlt models). Let $f : (Y, \Gamma) \rightarrow (X, \Delta)$ be a dlt blow-up of a log canonical pair (X, Δ) . Then (Y, Γ) is called a *dlt model* of (X, Δ) .

Next we introduce the definition of some models and the construction of the log MMP with scaling for \mathbb{Q} -factorial log canonical pairs. Our definition of models is slightly different from the traditional one in [KM] or [BCHM].

Definition 2.5 (cf. [B2, Definition 2.1] and [B2, Definition 2.2]). Let $\pi : X \rightarrow Z$ be a projective morphism from a normal variety to a variety and let (X, Δ) be a log canonical pair. Let $\pi' : X' \rightarrow Z$ be a projective morphism from a normal variety to Z and $\phi : X \dashrightarrow X'$ be a birational map over Z . Let E be the reduced ϕ^{-1} -exceptional divisor on X' , that is, $E = \sum E_j$ where E_j are ϕ^{-1} -exceptional prime divisors on X' . Then the pair $(X', \Delta' = \phi_*\Delta + E)$ is called a *log birational model* of (X, Δ) over Z . A log birational model (X', Δ') of (X, Δ) over Z is a *weak log canonical model* (*weak lc model*, for short) if

- $K_{X'} + \Delta'$ is nef over Z , and
- for any prime divisor D on X which is exceptional over X' , we have

$$a(D, X, \Delta) \leq a(D, X', \Delta').$$

A weak lc model (X', Δ') of (X, Δ) over Z is a *log minimal model* if

- X' is \mathbb{Q} -factorial, and
- the above inequality on discrepancies is strict.

A log minimal model (X', Δ') of (X, Δ) over Z is called a *good minimal model* if $K_{X'} + \Delta'$ is semi-ample over Z .

A log birational model (X', Δ') of (X, Δ) over Z is called a *Mori fiber space* if X' is \mathbb{Q} -factorial and there is a contraction $X' \rightarrow W$ over Z with $\dim W < \dim X'$ such that

- the relative Picard number $\rho(X'/W)$ is one and $K_{X'} + \Delta'$ is anti-ample over W , and
- for any prime divisor D over X , we have

$$a(D, X, \Delta) \leq a(D, X', \Delta')$$

and strict inequality holds if D is a divisor on X and exceptional over X' .

Definition 2.6 (The log MMP with scaling, cf. [B2, Definition 2.4], [F4, 4.4.11]). Let $\pi : X \rightarrow Z$ be a projective surjective morphism from a \mathbb{Q} -factorial normal variety to a variety and $(X, \Delta + C)$ be a log canonical pair such that $K_X + \Delta + C$ is π -nef, Δ is a boundary

\mathbb{R} -divisor and C is an effective \mathbb{R} -divisor. We set $X_0 = X$, $\Delta_{X_0} = \Delta$ and $C_{X_0} = C$ and set

$$\lambda_0 = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_0} + \Delta_{X_0} + \mu C_{X_0} \text{ is nef over } Z\}.$$

If $\lambda_0 = 0$, we have nothing to do. If $\lambda_0 > 0$, then there is a $(K_{X_0} + \Delta_{X_0})$ -negative extremal ray R_0 over Z such that $(K_{X_0} + \Delta_{X_0} + \lambda_0 C_{X_0}) \cdot R_0 = 0$ by [F3, Theorem 18.9]. Let $f_0 : X_0 \rightarrow V_0$ be the extremal contraction over Z given by R_0 . If f_0 is a divisorial contraction, then set $X_1 = V_0$, $\Delta_{X_1} = f_{0*}\Delta_{X_0}$ and $C_{X_1} = f_{0*}C_{X_0}$. If f_0 is a flipping contraction, then there is the flip $\phi : X_0 \dashrightarrow X_1$ of f_0 over Z by [B2, Corollary 1.2] or [HX, Corollary 1.8], and we set $\Delta_{X_1} = \phi_*\Delta_{X_0}$ and $C_{X_1} = \phi_*C_{X_0}$. By construction X_1 is \mathbb{Q} -factorial. We set

$$\lambda_1 = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_1} + \Delta_{X_1} + \mu C_{X_1} \text{ is nef over } Z\}.$$

Then we have $\lambda_1 \leq \lambda_0$. If $\lambda_1 = 0$, we stop the process. If $\lambda_1 > 0$, then there is a $(K_{X_1} + \Delta_{X_1})$ -negative extremal ray R_1 over Z such that $(K_{X_1} + \Delta_{X_1} + \lambda_1 C_{X_1}) \cdot R_1 = 0$. By repeating this process, we get a non-increasing sequence of nonnegative real numbers $\{\lambda_i\}_{i \geq 0}$ and a sequence of steps of the $(K_X + \Delta)$ -MMP over Z

$$(X = X_0, \Delta = \Delta_{X_0}) \dashrightarrow (X_1, \Delta_{X_1}) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_{X_i}) \dashrightarrow \cdots.$$

This log MMP is called the $(K_X + \Delta)$ -MMP over Z with scaling of C .

Remark 2.7. Let (X, Δ) be a log canonical pair and (X', Δ') be a log minimal model or a Mori fiber space of (X, Δ) . If the birational map $X \dashrightarrow X'$ is a birational contraction, then our definition of log minimal models and Mori fiber spaces coincides with the traditional one.

In [B2], log minimal models and Mori fiber spaces are supposed to be dlt. On the other hand we do not assume it in Definition 2.5. But the difference is intrinsically not important. Indeed, if a log canonical pair (X, Δ) has a log minimal model (X', Δ') as in Definition 2.5, any dlt model of (X', Δ') is also a log minimal model of (X, Δ) . If (X, Δ) has a Mori fiber space as in Definition 2.5, we can construct a Mori fiber space of (X, Δ) which is dlt by taking a dlt model of (X, Δ) and by running the log MMP with scaling. In this way, if (X, Δ) has a log minimal model (resp. Mori fiber space), we can construct a log minimal model (resp. Mori fiber space) of (X, Δ) which is dlt.

Next we introduce the definition of log canonical thresholds and pseudo-effective thresholds.

Definition 2.8 (Log canonical thresholds, cf. [HMX]). Let (X, Δ) be a log canonical pair and let $M \neq 0$ be an effective \mathbb{R} -Cartier \mathbb{R} -divisor.

Then the *log canonical threshold* of M with respect to (X, Δ) , denoted by $\text{lct}(X, \Delta; M)$, is

$$\text{lct}(X, \Delta; M) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tM) \text{ is log canonical}\}.$$

Definition 2.9 (Pseudo-effective thresholds). Let (X, Δ) be a log canonical pair and M be an effective \mathbb{R} -Cartier \mathbb{R} -divisor such that $K_X + \Delta + tM$ is pseudo-effective for some $t \geq 0$. Then the *pseudo-effective threshold* of M with respect to (X, Δ) , denoted by $\tau(X, \Delta; M)$, is

$$\tau(X, \Delta; M) = \inf\{t \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + tM \text{ is pseudo-effective}\}.$$

The following important theorems are proved by Hacon, McKernan and Xu [HMX]. In their paper, one is called the ACC for log canonical thresholds and another one is called the ACC for numerically trivial pairs.

Theorem 2.10 (ACC for log canonical thresholds, cf. [HMX, Theorem 1.1]). *Fix a positive integer n , a set $I \subset [0, 1]$ and a set $J \subset \mathbb{R}_{>0}$, where I and J satisfy the DCC. Let $\mathfrak{T}_n(I)$ be the set of log canonical pairs (X, Δ) , where X is a variety of dimension n and the coefficients of Δ belong to I . Then the set*

$$\{\text{lct}(X, \Delta; M) \mid (X, \Delta) \in \mathfrak{T}_n(I), \text{ the coefficients of } M \text{ belong to } J\}$$

satisfies the ACC.

Theorem 2.11 (ACC for numerically trivial pairs, cf. [HMX, Theorem D]). *Fix a positive integer n and a set $I \subset [0, 1]$, which satisfies the DCC.*

Then there is a finite set $I_0 \subset I$ with the following property:

If (X, Δ) is a log canonical pair such that

- (i) X is projective of dimension n ,*
- (ii) the coefficients of Δ belong to I , and*
- (iii) $K_X + \Delta$ is numerically trivial,*

then the coefficients of Δ belong to I_0 .

Finally we introduce the definition of log smooth models and two related results. Corollary 2.14 is a special kind of dlt blow-up used in this paper.

Definition 2.12 (Log smooth models, cf. [B2, Definition 2.3] and [B2, Remark 2.8]). Let (X, Δ) be a log canonical pair and $f : Y \rightarrow X$ be a log resolution of (X, Δ) . Let Γ be a boundary \mathbb{R} -divisor on Y such that (Y, Γ) is log smooth. Then (Y, Γ) is a *log smooth model* of (X, Δ) if we write

$$K_Y + \Gamma = f^*(K_X + \Delta) + F,$$

then

- (i) F is an effective f -exceptional divisor, and
- (ii) every f -exceptional prime divisor E satisfying $a(E, X, \Delta) > -1$ is a component of F and $\Gamma - \lfloor \Gamma \rfloor$.

By the definition, $\text{Supp } \Gamma = \text{Supp } f_*^{-1}\Delta \cup \text{Ex}(f)$ and the image of any lc center of (Y, Γ) on X is an lc center of (X, Δ) . For any f -exceptional prime divisor E , E is a component of F if and only if $a(E, X, \Delta) > -1$. When Δ is a \mathbb{Q} -divisor and $f : Y \rightarrow X$ is a log resolution of (X, Δ) , we can find a \mathbb{Q} -divisor Γ on Y such that (Y, Γ) is a log smooth model of (X, Δ) .

Lemma 2.13. *Let $\pi : X \rightarrow Z$ be a projective morphism from a normal variety to a variety. Let (X, Δ) be a log canonical pair. Then there is a log smooth model (Y, Γ) of (X, Δ) such that*

- (i) $\Gamma = \Gamma' + \Gamma''$, where $\Gamma' \geq 0$ and Γ'' is a reduced divisor,
- (ii) $(\pi \circ f)(\text{Supp } \Gamma'') \subsetneq Z$, and
- (iii) every lc center of $(Y, \Gamma - t\Gamma'')$ dominates Z for any $0 < t \leq 1$.

Proof. Replacing (X, Δ) with its log smooth model, we can assume that (X, Δ) is log smooth. For any lc center S of (X, Δ) not dominating Z , let $\pi_S : X_S \rightarrow X$ be the blow-up of X along S . Then X_S is a smooth variety and $\pi_S^{-1}(S)$ is a divisor on X_S . In particular it is a Cartier divisor on X_S . Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) such that Y is also a common resolution of all X_S , and construct a log smooth model (Y, Γ) of (X, Δ) . Let Γ'' be the reduced divisor such that Γ'' is the sum of all components of $\lfloor \Gamma \rfloor$ not dominating Z , and set $\Gamma' = \Gamma - \Gamma''$. Then Γ' and Γ'' satisfy the conditions (i) and (ii) of the lemma. We prove that Γ' and Γ'' satisfy the condition (iii) of the lemma.

Fix $0 < t \leq 1$ and let T be an lc center of $(Y, \Gamma - t\Gamma'')$. Since (Y, Γ) is lc, T is an lc center of (Y, Γ) and T is not contained in $\text{Supp } \Gamma''$. We prove that T dominates Z .

Suppose by contradiction that T does not dominates Z . Then $f(T)$ is an lc center of (X, Δ) not dominating Z and $(\pi_{f(T)})^{-1}(f(T))$ is a Cartier divisor on $X_{f(T)}$ by construction. Set $M = f^{-1}(f(T))$. Clearly we have $(\pi \circ f)(M) \subsetneq Z$, and M is a divisor because M is the support of the pullback of $(\pi_{f(T)})^{-1}(f(T))$. Moreover T is contained in a component of M because T is irreducible. Since $\text{Supp } \Gamma = \text{Supp } f_*^{-1}\Delta \cup \text{Ex}(f)$, we also have $M \subset \text{Supp } \Gamma$. Therefore T is contained in a component G of Γ such that $(\pi \circ f)(G) \subsetneq Z$. On the other hand, T is an irreducible component of the intersection of some divisors in $\lfloor \Gamma \rfloor$ because T is an lc center of the log smooth model (Y, Γ) . Since (Y, Γ) is log smooth,

the coefficient of G in Γ is one. Then T is contained in $\text{Supp } \Gamma''$ and we get a contradiction. Therefore T dominates Z and so we are done. \square

Let $\pi : X \rightarrow Z$ be a projective morphism from a normal variety to a variety and (X, Δ) be a log canonical pair. In the rest of this paper, the phrase “ $(X, \Delta = \Delta' + \Delta'')$ satisfies all the conditions of Lemma 2.13 with respect to π ” means that we can write $\Delta = \Delta' + \Delta''$ where Δ' and Δ'' satisfy the conditions (i), (ii) and (iii) of Lemma 2.13 with respect to π .

Corollary 2.14. *Let $\pi : X \rightarrow Z$ be a projective morphism of normal quasi-projective varieties and (X, Δ) be a log canonical pair. Then there is a dlt blow-up $f : (Y, \Gamma) \rightarrow (X, \Delta)$ such that $(Y, \Gamma = \Gamma' + \Gamma'')$ satisfies all the conditions of Lemma 2.13 with respect to $\pi \circ f$.*

Proof. Let $(Y, \Gamma = \Gamma' + \Gamma'') \rightarrow (X, \Delta)$ be a log smooth model of (X, Δ) as in Lemma 2.13. We run the $(K_Y + \Gamma)$ -MMP over X with scaling. By [B2, Theorem 3.4], we get a good minimal model $\phi : (Y, \Gamma) \dashrightarrow (Y', \Gamma_{Y'})$ over X . Let $f : Y' \rightarrow X$ be the induced morphism. Then f is a dlt blow-up of (X, Δ) . Set $\Gamma'_{Y'} = \phi_* \Gamma'$ and $\Gamma''_{Y'} = \phi_* \Gamma''$. Then we can check that $(Y, \Gamma_{Y'} = \Gamma'_{Y'} + \Gamma''_{Y'})$ satisfies all the conditions of Lemma 2.13 with respect to $\pi \circ f$ because $a(D, Y, \Gamma - t\Gamma'') \leq a(D, Y', \Gamma_{Y'} - t\Gamma''_{Y'})$ for any sufficiently small positive real number t and any prime divisor D over Y . Therefore $f : (Y', \Gamma_{Y'}) \rightarrow (X, \Delta)$ is the desired dlt blow-up. \square

3. MINIMAL MODEL PROGRAM

In this section we study the Minimal Model Program for any \mathbb{R} -Cartier \mathbb{R} -divisor D which is not necessarily a log canonical divisor. More precisely, we define a sequence of birational maps, which we call D -MMP, and construct the D -MMP under some assumptions.

Definition 3.1 (The D -MMP). Let X be a \mathbb{Q} -factorial normal projective variety and let D be an \mathbb{R} -divisor on X . Then a finite sequence of birational maps

$$\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$$

is a finitely many steps of the D -Minimal Model Program (D -MMP, for short) if

- (i) there exists a boundary \mathbb{R} -divisor Δ on X such that (X, Δ) is log canonical and ϕ is a finitely many steps of the $(K_X + \Delta)$ -MMP, and
- (ii) for any $0 \leq i < n$, the birational transform D_{X_i} of D on X_i , which is always \mathbb{R} -Cartier by the condition (i), is anti-ample

with respect to the extremal contraction $f_i : X_i \rightarrow V_i$, that is, $X_{i+1} = V_i$ or X_{i+1} is the flip of f_i .

An infinite sequence of birational maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots$$

is a sequence of steps of the D -MMP if $X \dashrightarrow X_i$ is a finitely many steps of the D -MMP for any i .

Remark 3.2. By the definition, all X_i are \mathbb{Q} -factorial. In (i) of the above definition, we can in fact find a boundary \mathbb{Q} -divisor instead of \mathbb{R} -divisor by [F4, Theorem 4.7.2 (3)].

Notation as above, suppose that D is a \mathbb{Q} -divisor and let $X_i \dashrightarrow X_{i+1}$ be a step of the D -MMP. Then it is a step of the $(K_X + \Delta)$ -MMP for some \mathbb{Q} -divisor Δ . Let $X_i \rightarrow V_i$ be the extremal contraction. Then we can write $X_{i+1} = \mathbf{Proj}(\mathcal{R}(X_i/V_i, D_{X_i}))$. Indeed, we can check that $X_{i+1} = \mathbf{Proj}(\mathcal{R}(X_i/V_i, K_{X_i} + \Delta_{X_i}))$ even if $X_i \rightarrow V_i$ is a divisorial contraction. By the cone theorem [F4, Theorem 4.5.2] and since D_{X_i} is anti-ample over V_i , $D_{X_i} \sim_{\mathbb{Q}, V_i} m(K_{X_i} + \Delta_{X_i})$ for some positive rational number m . Thus $X_{i+1} \simeq \mathbf{Proj}(\mathcal{R}(X_i/V_i, D_{X_i}))$.

Definition 3.3 (The D -MMP with scaling). Let X be a \mathbb{Q} -factorial normal projective variety and let D be an \mathbb{R} -divisor on X . Let A be an \mathbb{R} -divisor such that $D + A$ is nef. Then the sequence of birational maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots$$

is the D -MMP with scaling of A if

- (i) it is a sequence of steps of the D -MMP, and
- (ii) if we set

$$\lambda_i = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid D_{X_i} + \mu A_{X_i} \text{ is nef}\}$$

for any i , then $D_{X_i} + \lambda_i A_{X_i}$ is trivial with respect to the extremal contraction $X_i \rightarrow V_i$.

If divisors D and A on X are given and there is no confusion, we mean the D -MMP with scaling of A by

$$(X = X_0, \lambda_0) \dashrightarrow (X_1, \lambda_1) \dashrightarrow \cdots \dashrightarrow (X_i, \lambda_i) \dashrightarrow \cdots$$

where $\lambda_i = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid D_{X_i} + \mu A_{X_i} \text{ is nef}\}$.

Remark 3.4. Notation as above, let

$$(X = X_0, \lambda_0) \dashrightarrow (X_1, \lambda_1) \dashrightarrow \cdots \dashrightarrow (X_i, \lambda_i) \dashrightarrow \cdots$$

be a sequence of steps of the D -MMP with scaling of A . Pick an index $i \geq 0$ and a real number $t < \lambda_i$, which is not necessarily positive. By

the definition of the D -MMP with scaling, the sequence of birational maps $X_0 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1}$ is a finitely many steps of the $(D + tA)$ -MMP with scaling of $(1 - t)A$. If we set $\lambda'_j = (\lambda_j - t)/(1 - t)$ for any $0 \leq j \leq i$, then the $(D + tA)$ -MMP with scaling can be written

$$(X_0, \lambda'_0) \dashrightarrow \cdots \dashrightarrow (X_i, \lambda'_i) \dashrightarrow X_{i+1}.$$

In particular, if $t < \lambda_i$ for any i , then the above sequence of birational maps is the $(D + tA)$ -MMP with scaling of $(1 - t)A$

$$(X_0, \lambda'_0) \dashrightarrow \cdots \dashrightarrow (X_i, \lambda'_i) \dashrightarrow \cdots$$

where $\lambda'_i = (\lambda_i - t)/(1 - t)$ for any i .

If D is the log canonical divisor of a log canonical pair, then we can identify the D -MMP with the standard log MMP on the log canonical pair. Therefore Definition 3.1 is a generalization of the standard log MMP. Similarly, we can check that Definition 3.3 is a generalization of the standard definition of the log MMP with scaling.

Finally, we prove two results related to the D -MMP with scaling.

Lemma 3.5. *Let $\pi : X \rightarrow Z$ be a projective surjective morphism from a normal projective variety to a \mathbb{Q} -factorial normal projective variety with connected fibers, and let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor. Suppose that $(X, \Delta = \Delta' + \Delta'')$ satisfies all the conditions of Lemma 2.13 with respect to π . Suppose in addition that $K_X + \Delta \sim_{\mathbb{Q}} \pi^*D$ and $\Delta'' \sim_{\mathbb{Q}} \pi^*E$ for a \mathbb{Q} -divisor D and an effective \mathbb{Q} -divisor E on Z respectively. Let A be a big semi-ample \mathbb{Q} -divisor on Z such that $A + E$ is also semi-ample and $D + A$ is nef. Then there is a sequence of birational maps of the D -MMP with scaling of A*

$$(Z = Z_0, \lambda_0) \dashrightarrow \cdots \dashrightarrow (Z_i, \lambda_i) \dashrightarrow \cdots$$

such that the D -MMP terminates or $\lim_{i \rightarrow \infty} \lambda_i = 0$ when the D -MMP does not terminate. In particular, we always have $\lim_{i \rightarrow \infty} \lambda_i = 0$ when D is pseudo-effective.

Lemma 3.6. *Let $\pi : X \rightarrow Z$ be a projective surjective morphism of \mathbb{Q} -factorial normal projective varieties with connected fibers, and let (X, Δ) be a log canonical pair where Δ is a \mathbb{Q} -divisor. Suppose that $(X, 0)$ is Kawamata log terminal and there is a \mathbb{Q} -divisor D on Z such that $K_X + \Delta \sim_{\mathbb{Q}} \pi^*D$. Let A be an effective \mathbb{Q} -divisor on Z such that $D + A$ is nef and $(X, \Delta + \pi^*A)$ is log canonical. Suppose that there is a sequence of birational maps of the D -MMP with scaling of A*

$$(Z = Z_0, \lambda_0) \dashrightarrow \cdots \dashrightarrow (Z_i, \lambda_i) \dashrightarrow \cdots$$

with the corresponding numbers λ_i defined in Definition 3.3. We set $X_0 = X$ and $\Delta_{X_0} = \Delta$. Then we have the following diagram

$$\begin{array}{ccccccc} (X_0, \Delta_{X_0}) & \dashrightarrow & \cdots & \dashrightarrow & (X_{k_1}, \Delta_{X_{k_1}}) & \dashrightarrow & \cdots & \dashrightarrow & (X_{k_i}, \Delta_{X_{k_i}}) & \dashrightarrow & \cdots \\ \pi = \pi_0 \downarrow & & & & \pi_1 \downarrow & & & & \pi_i \downarrow & & \\ (Z_0, \lambda_0) & \dashrightarrow & \cdots & \dashrightarrow & (Z_1, \lambda_1) & \dashrightarrow & \cdots & \dashrightarrow & (Z_i, \lambda_i) & \dashrightarrow & \cdots \end{array}$$

such that

- (i) for any i , π_i is projective and surjective with connected fibers,
- (ii) the upper horizontal sequence of birational maps is a sequence of steps of the $(K_X + \Delta)$ -MMP with scaling of π^*A such that if we set $k_0 = 0$ and

$$\lambda'_j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_j} + \Delta_{X_j} + \mu(\pi^*A)_{X_j} \text{ is nef}\},$$

where $(\pi^*A)_{X_j}$ is the birational transform of π^*A on X_j , then $\lambda'_j = \lambda_i$ for any $k_i \leq j < k_{i+1}$, and

- (iii) for any two indices $i < i'$ and any \mathbb{Q} -divisor B on Z_i , we have $(\pi_i^*B)_{X_{k_{i'}}} = \pi_{i'}^*B_{Z_{i'}}$.

In particular, $K_{X_{k_i}} + \Delta_{X_{k_i}} \sim_{\mathbb{Q}} \pi_i^*D_{Z_i}$ for any i and the $(K_X + \Delta)$ -MMP with scaling of π^*A terminates if and only if the D -MMP with scaling of A terminates.

Proof of Lemma 3.5. Fix a strictly decreasing infinite sequence $\{a_n\}_{n \geq 1}$ of rational numbers such that $0 < a_n < 1$ for any n and $\lim_{n \rightarrow \infty} a_n = 0$. By the condition (iii) of Lemma 2.13, any lc center of $(X, \Delta - a_n \Delta'')$ dominates Z for any n . We also have $K_X + \Delta - a_n \Delta'' \sim_{\mathbb{Q}} \pi^*(D - a_n E)$ by the hypothesis. By [FG1, Corollary 3.2], there are \mathbb{Q} -divisors Ψ_n on Z such that all (Z, Ψ_n) are klt and $D - a_n E \sim_{\mathbb{Q}} K_Z + \Psi_n$. Fix a sufficiently general semi-ample \mathbb{Q} -divisor $A' \sim_{\mathbb{Q}} A + E$ satisfying that $(Z, \Psi_n + A')$ is klt for any $n \geq 1$. Similarly, fix a sufficiently general semi-ample \mathbb{Q} -divisor $A'' \sim_{\mathbb{Q}} A$ satisfying that $(Z, \Psi_n + A' + A'')$ is klt for any $n \geq 1$. Then $(Z, \Psi_n + a_n A' + t A'')$ is also klt and

$$\begin{aligned} D + (t + a_n)A &= (D - a_n E) + a_n(A + E) + tA \\ (\spadesuit) \quad &\sim_{\mathbb{Q}} K_Z + \Psi_n + a_n A' + t A'' \end{aligned}$$

for any $0 \leq t \leq 1$ and $n \geq 1$. Note that A' is big since A is big and E is effective.

Since $K_Z + \Psi_1 + a_1 A' + (1 - a_1) A'' \sim_{\mathbb{Q}} D + A$ is nef, we can run the $(K_Z + \Psi_1 + a_1 A')$ -MMP with scaling of $(1 - a_1) A''$. By [BCHM, Corollary 1.4.2], this log MMP terminates with a good minimal model

or a Mori fiber space

$$\phi : Z = Z_0 \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_{k_1} = Z'$$

of $(Z, \Psi_1 + a_1 A')$. It is also a finitely many steps of the $(D + a_1 A)$ -MMP because $D + a_1 A \sim_{\mathbb{Q}} K_Z + \Psi_1 + a_1 A'$.

For any $i \geq 0$, we set

$$\lambda_i = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{Z_i} + (\Psi_1)_{Z_i} + a_1 A'_{Z_i} + \mu(1 - a_1)A''_{Z_i} \text{ is nef}\}$$

where $(\Psi_1)_{Z_i}$ is the birational transform of Ψ_1 on Z_i . We note that $\lambda_{k_1-1} > 0$ by the definition of the log MMP with scaling. By the above (\spadesuit) , for any $0 \leq i < k_1$, $D_{Z_i} + (a_1 + \lambda_i(1 - a_1))A_{Z_i}$ is nef and trivial with respect to the extremal contraction of the $(K_Z + \Psi_1 + a_1 A')$ -MMP. Since $D_{Z_i} + a_1 A_{Z_i}$ is anti-ample with respect to the extremal contraction, D_{Z_i} is anti-ample with respect to the extremal contraction for any $0 \leq i < k_1$. Moreover, if we set

$$\lambda'_i = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid D_{Z_i} + \mu A_{Z_i} \text{ is nef}\}$$

for any $0 \leq i < k_1$, then $\lambda'_i = a_1 + \lambda_i(1 - a_1)$ by the above discussion. Therefore ϕ is a finitely many steps of the D -MMP with scaling of A (see Definition 3.3). Pick a rational number $t \leq a_1$. Then we have $t < \lambda'_{k_1-1}$ since $a_1 < \lambda'_{k_1-1}$. By Remark 3.4, ϕ is a finitely many steps of the $(D + tA)$ -MMP with scaling of $(1 - t)A$ for any $0 \leq t \leq a_1$. Since $K_Z + \Psi_n + a_n A' \sim_{\mathbb{Q}} D + a_n A$, $A'' \sim_{\mathbb{Q}} A$ and $\{a_n\}_{n \geq 1}$ is a strictly decreasing sequence, we see that ϕ is also a finitely many steps of the $(K_Z + \Psi_n + a_n A')$ -MMP with scaling of $(1 - a_n)A''$ for any $n \geq 1$.

If $(Z', (\Psi_1)_{Z'} + a_1 A'_{Z'})$ is a Mori fiber space, then the D -MMP with scaling terminates and we stop the process. If $(Z', (\Psi_1)_{Z'} + a_1 A'_{Z'})$ is a good minimal model of $(Z, \Psi_1 + a_1 A')$, then $\lambda_{k_1} = 0$. By the above (\spadesuit) we have

$$\begin{aligned} K_{Z'} + (\Psi_1)_{Z'} + a_1 A'_{Z'} &\sim_{\mathbb{Q}} D_{Z'} + a_1 A_{Z'} \\ &\sim_{\mathbb{Q}} K_{Z'} + (\Psi_2)_{Z'} + a_2 A'_{Z'} + (a_1 - a_2)A''_{Z'}, \end{aligned}$$

and thus $D_{Z'} + a_1 A_{Z'}$ and $K_{Z'} + (\Psi_2)_{Z'} + a_2 A'_{Z'} + (a_1 - a_2)A''_{Z'}$ are nef. Since ϕ is a finitely many steps of the $(K_Z + \Psi_2 + a_2 A')$ -MMP, $(Z', (\Psi_2)_{Z'} + a_2 A'_{Z'})$ is klt. Then we can run the $(K_{Z'} + (\Psi_2)_{Z'} + a_2 A'_{Z'})$ -MMP with scaling $(a_1 - a_2)A''_{Z'}$. By [BCHM, Corollary 1.4.2], this log MMP terminates with a good minimal model or a Mori fiber space

$$\psi : Z' = Z_{k_1} \dashrightarrow Z_{k_1+1} \dashrightarrow \cdots \dashrightarrow Z_{k_2} = Z''$$

of $(Z', (\Psi_2)_{Z'} + a_2 A'_{Z'})$. By the same discussion as above, we can check that $\psi \circ \phi : Z \dashrightarrow Z''$ is a finitely many steps of the D -MMP with scaling

of A and also a finitely many steps of the $(K_Z + \Psi_n + a_n A')$ -MMP with scaling of $(1 - a_n)A''$ for any $n \geq 2$.

By repeating the above discussions, we get a sequence of birational maps

$$Z = Z_0 \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_{k_i} \dashrightarrow \cdots$$

such that

- for any $i \geq 1$, the birational map $Z \dashrightarrow Z_{k_i}$ is a finitely many steps of the $(K_Z + \Psi_i + a_i A')$ -MMP with scaling of $(1 - a_i)A''$ to a good minimal model or a Mori fiber space,
- the whole sequence of birational maps $Z \dashrightarrow \cdots \dashrightarrow Z_j \dashrightarrow \cdots$ is a sequence of steps of the D -MMP with scaling of A , and
- If we set

$$\lambda_j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid D_{Z_j} + \mu A_{Z_j} \text{ is nef}\},$$

then $\lambda_{k_i} \leq a_i$.

The third condition follows from the fact that $D_{Z_{k_i}} + a_i A_{Z_{k_i}}$ is nef. If the D -MMP does not terminate, then $\lim_{i \rightarrow \infty} \lambda_i \leq \lim_{i \rightarrow \infty} a_i = 0$ by the definition of $\{a_n\}_{n \geq 1}$. Therefore $\lim_{i \rightarrow \infty} \lambda_i = 0$ and thus the above D -MMP with scaling of A satisfies all the conditions of the lemma. So we are done. \square

Poof of Lemma 3.6. Set $\pi = \pi_0$ and let $f : Z_0 \rightarrow V_0$ be the extremal contraction. Note that $K_{X_0} + \Delta_{X_0} + \pi_0^* A$ is nef since $D + A$ is nef. By the definition of D -MMP and the cone theorem [F4, Theorem 4.5.2], there is a general ample \mathbb{Q} -divisor H on Z_0 such that $D + H \sim_{\mathbb{Q}, V_0} 0$. Therefore $K_{X_0} + \Delta_{X_0} + \pi_0^* H \sim_{\mathbb{Q}, V_0} 0$. By [B2, Theorem 1.1] and [B2, Theorem 4.1 (iii)], the $(K_{X_0} + \Delta_{X_0})$ -MMP over V_0 with scaling of an ample divisor terminates with a good minimal model

$$\phi : (X_0, \Delta_{X_0}) \dashrightarrow (X_1, \Delta_{X_1}) \dashrightarrow \cdots \dashrightarrow (X_{k_1} = X', \Delta_{X_{k_1}} = \Delta_{X'})$$

over V_0 . Then we can check that ϕ is a finitely many steps of the $(K_{X_0} + \Delta_{X_0})$ -MMP with scaling of $\pi_0^* A$ and if we set

$$\lambda'_j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_j} + \Delta_{X_j} + \mu(\pi_0^* A)_{X_j} \text{ is nef}\}$$

for any $0 \leq j \leq k_1$, then $\lambda'_0 = \lambda'_1 = \cdots = \lambda'_{k_1-1} = \lambda_0$ (see, for example, the proof of [H, Proposition 4.1]).

Since $K_{X'} + \Delta_{X'}$ is semi-ample over V_0 , there is a natural morphism $X' \rightarrow Z' = \mathbf{Proj}(\mathcal{R}(X'/V_0, K_{X'} + \Delta_{X'}))$ over V_0 . By construction we have $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}, Z'} 0$. Now $Z_1 = \mathbf{Proj}(\mathcal{R}(Z_0/V_0, D))$ by Remark

3.2, and for any large and divisible positive integer m ,

$$\begin{aligned}\mathcal{R}(Z_0/V_0, mD) &\simeq \mathcal{R}(X_0/V_0, m(K_{X_0} + \Delta_{X_0})) \\ &\simeq \mathcal{R}(X'/V_0, m(K_{X'} + \Delta_{X'}))\end{aligned}$$

as sheaves of graded \mathcal{O}_{V_0} -algebra. Therefore we have $Z' \simeq Z_1$. We put $\pi_1 : X' \rightarrow Z_1 \simeq Z'$. Then we see that π_1 has connected fibers by taking a common resolution of ϕ . We also see that $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}} \pi_1^* D_{Z_1}$ because $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}, Z_1} 0$ and $K_{X_0} + \Delta_{X_0} \sim_{\mathbb{Q}} \pi_0^* D$. We can check that $(X', 0)$ is klt. Indeed, $(X_0, 0)$ is klt by hypothesis. Therefore $(X_0, t\Delta_{X_0})$ is also klt for any $0 \leq t < 1$. For any $t < 1$ sufficiently close to one, ϕ is a finitely many steps of the $(K_{X_0} + t\Delta_{X_0})$ -MMP. Then $(X', t\Delta_{X'})$ is klt for some $t < 1$ and thus $(X', 0)$ is klt because X' is \mathbb{Q} -factorial.

We prove $(\pi_0^* B)_{X'} = \pi_1^* B_{Z_1}$ for any \mathbb{Q} -divisor B on Z_0 . First we prove $(\pi_0^* B)_{X'} \sim_{\mathbb{Q}} \pi_1^* B_{Z_1}$, and after that we prove $(\pi_0^* B)_{X'} = \pi_1^* B_{Z_1}$. Recall that $f : Z_0 \rightarrow V_0$ is the extremal contraction of the D -MMP. Let $f_1 : Z_1 \rightarrow V_0$ be the induced morphism. By construction, there is a rational number r and \mathbb{Q} -Cartier \mathbb{Q} -divisor G on V_0 satisfying $B - rD \sim_{\mathbb{Q}} f^* G$. Then $\pi_0^* B - r(K_{X_0} + \Delta_{X_0}) \sim_{\mathbb{Q}} \pi_0^* f^* G$. By taking the birational transform on X' , we obtain $(\pi_0^* B)_{X'} - r\pi_1^* D_{Z_1} \sim_{\mathbb{Q}} \pi_1^* f_1^* G$ because $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}} \pi_1^* D_{Z_1}$. Since $B_{Z_1} - rD_{Z_1} \sim_{\mathbb{Q}} f_1^* G$, we see that $(\pi_0^* B)_{X'} \sim_{\mathbb{Q}} \pi_1^* B_{Z_1}$. With this relation we prove $(\pi_0^* B)_{X'} = \pi_1^* B_{Z_1}$ as divisors. We note that B or $-B$ is nef over V_0 because the relative Picard number $\rho(Z_0/V_0)$ is one. Let $p : \tilde{Z} \rightarrow Z_0$ and $p' : \tilde{Z} \rightarrow Z_1$ be a common resolution of $Z_0 \dashrightarrow Z_1$, and let $q : \tilde{X} \rightarrow X_0$ and $q' : \tilde{X} \rightarrow X'$ be a common resolution of $\phi : X_0 \dashrightarrow X'$ such that the induced map $h : \tilde{X} \rightarrow \tilde{Z}$ is a morphism. We set $F = p^* B - p'^* B_{Z_1}$. Then F or $-F$ is effective by the negativity lemma. Moreover, by construction, we have

$$\begin{aligned}(\pi_0^* B)_{X'} - \pi_1^* B_{Z_1} &= q'_* q^* \pi_0^* B - q'_* q'^* \pi_1^* B_{Z_1} \\ &= q'_* (h^* p^* B) - q'_* (h^* p'^* B_{Z_1}) = q'_* h^* F.\end{aligned}$$

On the other hand, since $(\pi_0^* B)_{X'} \sim_{\mathbb{Q}} \pi_1^* B_{Z_1}$, we have $q'_* h^* F \sim_{\mathbb{Q}} 0$. Then $q'_* h^* F = 0$ because F or $-F$ is effective. In this way, we see that $(\pi_0^* B)_{X'} = \pi_1^* B_{Z_1}$ as divisors.

Now we have $(\pi_0^* A)_{X'} = \pi_1^* A_{Z_1}$. Since $(X', \Delta_{X'} + \lambda_1(\pi_0^* A)_{X'})$ is lc, $(X', \Delta_{X'} + \lambda_1 \pi_1^* A_{Z_1})$ is lc. Therefore we can apply the above arguments to $\pi_1 : (X_{k_1}, \Delta_{X_{k_1}}) = (X', \Delta_{X'}) \rightarrow Z_1$ and $\lambda_1 A_{Z_1}$. By repeating these

arguments, we have the following diagram

$$\begin{array}{ccccccc}
 (X_0, \Delta_{X_0}) & \dashrightarrow & \cdots & \dashrightarrow & (X_{k_1}, \Delta_{X_{k_1}}) & \dashrightarrow & \cdots & \dashrightarrow & (X_{k_i}, \Delta_{X_{k_i}}) & \dashrightarrow & \cdots \\
 \pi_0 \downarrow & & & & \pi_1 \downarrow & & & & \pi_i \downarrow & & \\
 (Z_0, \lambda_0) & \dashrightarrow & \cdots & \dashrightarrow & (Z_1, \lambda_1) & \dashrightarrow & \cdots & \dashrightarrow & (Z_i, \lambda_i) & \dashrightarrow & \cdots
 \end{array}$$

such that

- for any i , π_i is projective and surjective with connected fibers,
- the upper horizontal sequence of birational maps is a sequence of steps of the $(K_{X_0} + \Delta_{X_0})$ -MMP with scaling of π_0^*A such that if we set $k_0 = 0$ and

$$\lambda'_j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_j} + \Delta_{X_j} + \mu(\pi_0^*A)_{X_j} \text{ is nef}\},$$

then $\lambda'_j = \lambda_i$ for any $k_i \leq j < k_{i+1}$, and

- $(\pi_i^*B)_{X_{k_{i+1}}} = \pi_{i+1}^*B_{Z_{i+1}}$ for any i and any \mathbb{Q} -divisor B on Z_i .

Pick any two indices $i < i'$ and \mathbb{Q} -divisor B on Z_i . Then we can check that $(\pi_i^*B)_{X_{k_{i'}}} = \pi_{i'}^*B_{Z_{i'}}$ by the induction on $i' - i$. Therefore the diagram satisfies all the conditions of the lemma. \square

4. REFORMULATION OF THE MAIN RESULT

In this section we reformulate Theorem 1.2 and collect some results to prove Theorem 1.2. We start with the following lemma. It is well known to the experts, but we give the proof for the reader's convenience.

Lemma 4.1. *Fix a positive integer d_0 , and assume existence of a good minimal model or a Mori fiber space for all d_0 -dimensional projective Kawamata log terminal pairs with boundary \mathbb{Q} -divisors.*

Then there is a good minimal model or a Mori fiber space for all d -dimensional projective Kawamata log terminal pairs with boundary \mathbb{Q} -divisors such that $d \leq d_0$.

Proof. Let (X, Δ) be a d -dimensional projective klt pair such that Δ is a \mathbb{Q} -divisor. By taking a dlt blow-up and replacing (X, Δ) if necessary, we may assume that X is \mathbb{Q} -factorial. Moreover we may assume that $K_X + \Delta$ is pseudo-effective because otherwise we can find a Mori fiber space of (X, Δ) by running the $(K_X + \Delta)$ -MMP with scaling. By replacing (X, Δ) with its log smooth model, we may assume that (X, Δ) is log smooth. Let Y be the product of X and a $(d_0 - d)$ -dimensional abelian variety A , and let $p : Y \rightarrow X$ (resp. $q : Y \rightarrow A$) be the projection to X (resp. A). Set $\Gamma = p^*\Delta$. By construction, we have $K_Y + \Gamma = p^*(K_X + \Delta)$ and that (Y, Γ) is klt.

Run the $(K_Y + \Gamma)$ -MMP with scaling of an ample divisor. By the hypothesis, the log MMP must terminate with a good minimal model. We also see that this log MMP is the log MMP over A because A is an abelian variety, which contains no rational curves. Therefore we get a good minimal model $(Y, \Gamma) \dashrightarrow (Y', \Gamma_{Y'})$ over A . Let $q' : Y' \rightarrow A$ be the induced morphism. Let F and F' be the general fiber of q and q' respectively, and let $(\tilde{F}, \Delta_{\tilde{F}})$ be a dlt blow-up of $(F', \Gamma_{Y'}|_{F'})$. Then $(\tilde{F}, \Delta_{\tilde{F}})$ is a \mathbb{Q} -factorial klt pair. By construction, we have $F = X$ and $K_F + \Gamma|_F = K_X + \Delta$. Moreover we can check that $(\tilde{F}, \Delta_{\tilde{F}})$ is a good minimal model of $(X = F, \Delta = \Gamma|_F)$ because $(Y', \Gamma_{Y'})$ is a good minimal model of (Y, Γ) over A . So we are done. \square

Now we reformulate Theorem 1.2. We can easily check that Theorem 1.2 follows from Lemma 4.1 and the following theorem.

Theorem 4.2. *Fix a nonnegative integer d_0 , and assume existence of a good minimal model or a Mori fiber space for all d -dimensional projective Kawamata log terminal pairs with boundary \mathbb{Q} -divisors such that $d \leq d_0$.*

Let $\pi : X \rightarrow Z$ be a projective surjective morphism of normal projective varieties such that $\dim Z \leq d_0$, and let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor. Suppose that $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$.

Then (X, Δ) has a good minimal model or a Mori fiber space.

From now on we devote to prove Theorem 4.2. Proposition 4.3 below is a special case of Theorem 4.2.

Proposition 4.3. *Fix a positive integer d , and assume existence of a good minimal model or a Mori fiber space for all projective Kawamata log terminal pairs of dimension d with boundary \mathbb{Q} -divisors.*

Let $\pi : X \rightarrow Z$ be a projective surjective morphism of normal projective varieties such that $\dim Z = d$. Let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor and every lc center of (X, Δ) dominates Z . Suppose that $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$.

Then (X, Δ) has a good minimal model or a Mori fiber space.

Proof. We can prove this by the same arguments as in the proof of [BH, Proposition 3.3]. We write details for the reader's convenience.

By taking the Stein factorization of π , we may assume that π has connected fibers. We may also assume that $K_X + \Delta$ is pseudo-effective because otherwise we can find a Mori fiber space of (X, Δ) by taking a dlt model and by running the $(K_X + \Delta)$ -MMP with scaling. By [FG1, Corollary 3.2], there is a \mathbb{Q} -divisor Ψ on Z such that (Z, Ψ) is klt and $K_X + \Delta \sim_{\mathbb{Q}} \pi^*(K_Z + \Psi)$. Then $K_Z + \Psi$ is pseudo-effective (cf. [N,

II 5.6 Lemma]). By the hypothesis, there is a good minimal model $\phi : (Z, \Psi) \dashrightarrow (Z', \Psi_{Z'})$ of (Z, Ψ) .

Let $f : W \rightarrow Z$ and $f' : W \rightarrow Z'$ be a common resolution of ϕ and let $g : (Y, \Gamma) \rightarrow (X, \Delta)$ be a log smooth model such that the induced map $h : Y \dashrightarrow W$ is a morphism. Then we see that $f' \circ h : Y \rightarrow Z'$ has connected fibers. Moreover we have $K_Y + \Gamma = g^*(K_X + \Delta) + E$ for an effective g -exceptional divisor E and $f^*(K_Z + \Psi) = f'^*(K_{Z'} + \Psi_{Z'}) + F$ for an effective f' -exceptional divisor F . Then

$$\begin{aligned} K_Y + \Gamma &= g^*(K_X + \Delta) + E \sim_{\mathbb{Q}} g^* \pi^*(K_Z + \Psi) + E \\ &= h^* f^*(K_Z + \Psi) + E = h^* f'^*(K_{Z'} + \Psi_{Z'}) + h^* F + E. \end{aligned}$$

We run the $(K_Y + \Gamma)$ -MMP over Z' with scaling of an ample divisor

$$Y \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_i \dashrightarrow \cdots$$

Pick an open set U of Z such that the restriction of ϕ to U is an isomorphism $\phi|_U : U \rightarrow \phi(U)$ and the codimension of $Z' \setminus \phi(U)$ in Z' is at least two. By shrinking U if necessary, we can assume that F is mapped into $Z' \setminus \phi(U)$. Set $V = (\pi \circ g)^{-1}(U)$. Since $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$ and by the definition of log smooth models, we see that $(\pi^{-1}(U), \Delta|_{\pi^{-1}(U)})$ is a weak lc model of $(V, \Gamma|_V)$ over U with relatively trivial log canonical divisor. Since $U \simeq \phi(U)$, the $(K_Y + \Gamma)$ -MMP over Z' must terminate over $\phi(U)$ and E is eventually contracted over $\phi(U)$.

By the above facts, we see that $(h^*F)_{Y_i} + E_{Y_i}$ is mapped into $Z' \setminus \phi(U)$ for any $i \gg 0$. In particular $(h^*F)_{Y_i} + E_{Y_i}$ is a very exceptional divisor over Z' (cf. [B2, Definition 3.1]). Moreover, by the definition of the log MMP with scaling, $K_{Y_i} + \Gamma_{Y_i} \sim_{\mathbb{Q}, Z'} (h^*F)_{Y_i} + E_{Y_i}$ is the limit of movable divisors over Z' for any $i \gg 0$. Then $(h^*F)_{Y_i} + E_{Y_i} = 0$ by [B2, Lemma 3.3]. Therefore $K_{Y_i} + \Gamma_{Y_i} \sim_{\mathbb{Q}, Z'} 0$ for some i and thus $K_{Y_i} + \Gamma_{Y_i}$ is \mathbb{Q} -linearly equivalent to the pullback of $K_{Z'} + \Psi_{Z'}$. Since $K_{Z'} + \Psi_{Z'}$ is semi-ample, $K_{Y_i} + \Gamma_{Y_i}$ is also semi-ample. Therefore (Y_i, Γ_{Y_i}) is a good minimal model of (Y, Γ) . Since (Y, Γ) is a log smooth model of (X, Δ) , (X, Δ) also has a good minimal model. So we are done. \square

We can prove the following proposition by using [BCHM, Corollary 1.4.2] and the same discussion as in the proof of Proposition 4.3.

Proposition 4.4. *Let $\pi : X \rightarrow Z$ be a projective surjective morphism of \mathbb{Q} -factorial normal projective varieties with connected fibers. Let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor and every lc center of (X, Δ) dominates Z . Suppose that $K_X + \Delta \sim_{\mathbb{Q}} \pi^* D$ for a \mathbb{Q} -divisor D on Z .*

If D is big, then (X, Δ) has a good minimal model.

Finally we prove a reduction lemma, which plays a key role in the proof of Theorem 4.2.

Lemma 4.5. *To prove Theorem 4.2, we may assume the following conditions about $\pi : (X, \Delta) \rightarrow Z$.*

- (i) π has connected fibers, $(X, 0)$ is a \mathbb{Q} -factorial Kawamata log terminal pair, and Z is \mathbb{Q} -factorial,
- (ii) $(X, \Delta = \Delta' + \Delta'')$ satisfies all the conditions of Lemma 2.13 with respect to π ,
- (iii) $K_X + \Delta \sim_{\mathbb{Q}} \pi^* D$ and $\Delta'' \sim_{\mathbb{Q}} \pi^* E$, where D is a pseudo-effective \mathbb{Q} -divisor and E is an effective \mathbb{Q} -divisor on Z , and
- (iv) there is a \mathbb{Q} -divisor A on X such that $K_X + \Delta + \delta A$ is movable for any sufficiently small $\delta > 0$.

Proof. By taking a dlt blow-up and by replacing (X, Δ) if necessary, we can assume that X is \mathbb{Q} -factorial and $(X, 0)$ is klt. We may also assume that $K_X + \Delta$ is pseudo-effective because otherwise we can find a Mori fiber space of (X, Δ) by running the $(K_X + \Delta)$ -MMP with scaling. We note that existence of a good minimal model of (X, Δ) is equivalent to existence of a weak lc model of (X, Δ) with semi-ample log canonical divisor (see [B2, Corollary 3.7]). By taking the Stein factorization of π and by Corollary 2.14, we may assume that π has connected fibers and the condition (ii) of the lemma holds.

Next we show that we can assume $\Delta'' \sim_{\mathbb{Q}, Z} 0$ to prove Theorem 4.2. Since $K_X + \Delta$ is pseudo-effective and Δ'' is vertical over Z , we see that $K_X + \Delta'$ is pseudo-effective over Z . We run the $(K_X + \Delta')$ -MMP over Z with scaling of an ample divisor. By [B2, Theorem 1.1], this log MMP terminates with a good minimal model $\phi : (X, \Delta') \rightarrow (X', \Delta'_{X'})$ of (X, Δ') over Z because $K_X + \Delta' + \Delta'' \sim_{\mathbb{Q}, Z} 0$. Set $\Delta''_{X'} = \phi_* \Delta''$ and $\Delta_{X'} = \Delta'_{X'} + \Delta''_{X'} = \phi_* \Delta$. Then $(X', \Delta_{X'} = \Delta'_{X'} + \Delta''_{X'})$ satisfies all the conditions of Lemma 2.13 with respect to the morphism $X' \rightarrow Z$. Let $\pi' : X' \rightarrow Z'$ be the Stein factorization of the projective morphism induced by $K_{X'} + \Delta'_{X'}$ over Z . Then π' has connected fibers. Since $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}, Z} 0$ and $\Delta''_{X'}$ is vertical over Z , there is an open set U in Z such that $K_{X'} + \Delta'_{X'}$ is trivial over U . Therefore the morphism $Z' \rightarrow Z$ is birational, and thus $(X', \Delta_{X'} = \Delta'_{X'} + \Delta''_{X'})$ satisfies all the conditions of Lemma 2.13 with respect to π' . Moreover we have $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}, Z'} 0$ and $K_{X'} + \Delta'_{X'} \sim_{\mathbb{Q}, Z'} 0$. Therefore $\Delta''_{X'} \sim_{\mathbb{Q}, Z'} 0$. Since ϕ is a birational contraction and both $K_X + \Delta$ and $K_{X'} + \Delta_{X'}$ are \mathbb{Q} -linearly equivalent to the pullback of the same divisor on Z , (X, Δ) has a weak lc model with semi-ample log canonical divisor if $(X', \Delta_{X'})$ has a weak lc model with semi-ample log canonical divisor. In this way, by replacing $\pi : (X, \Delta) \rightarrow Z$ with $\pi' : (X', \Delta_{X'}) \rightarrow Z'$, we may assume

that $\Delta'' \sim_{\mathbb{Q}, Z} 0$. Then there is an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor E on Z such that $\Delta'' \sim_{\mathbb{Q}} \pi^* E$.

Now we can prove that we may assume the condition (i) of the lemma to prove Theorem 4.2. We may assume that π has connected fibers and $(X, 0)$ is \mathbb{Q} -factorial klt by the above arguments. Let D be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on Z such that $K_X + \Delta \sim_{\mathbb{Q}} \pi^* D$, and let E be an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor on Z such that $\Delta'' \sim_{\mathbb{Q}} \pi^* E$. By the condition (ii) of this lemma and the condition (iii) of Lemma 2.13, every lc center of (X, Δ') dominates Z . Since $K_X + \Delta' \sim_{\mathbb{Q}, Z} 0$, by [FG1, Corollary 3.2], there exists a klt pair on Z . Let $f : \tilde{Z} \rightarrow Z$ be a dlt blow-up of the klt pair. By construction \tilde{Z} is \mathbb{Q} -factorial and Z and \tilde{Z} are isomorphic in codimension one. Let $g : (Y, \Gamma') \rightarrow (X, \Delta')$ be a log smooth model of (X, Δ') such that the induced map $h : Y \dashrightarrow \tilde{Z}$ is a morphism. Then $\pi \circ g = f \circ h$ and we can write

$$K_Y + \Gamma' = g^*(K_X + \Delta') + F \sim_{\mathbb{Q}} (f \circ h)^*(D - E) + F$$

where F is an effective g -exceptional divisor which contains every g -exceptional prime divisor whose discrepancy with respect to (X, Δ') is greater than -1 . We run the $(K_Y + \Gamma')$ -MMP with scaling over \tilde{Z} . Since \tilde{Z} and Z are isomorphic in codimension one, by the same argument as in the proof of Proposition 4.3, we obtain a good minimal model $\phi : (Y, \Gamma') \dashrightarrow (Y', \Gamma'_{Y'})$ over \tilde{Z} . Let $h' : Y' \rightarrow \tilde{Z}$ be the induced morphism. Then Y' is \mathbb{Q} -factorial, $(Y', 0)$ is klt and h' has connected fibers. We also have $F_{Y'} = 0$ and $K_{Y'} + \Gamma'_{Y'} \sim_{\mathbb{Q}} (f \circ h')^*(D - E)$ by construction (see the proof of Proposition 4.3). Set

$$\Gamma''_{Y'} = \phi_* g^* \Delta'' \sim_{\mathbb{Q}} (f \circ h')^* E \quad \text{and} \quad \Gamma_{Y'} = \Gamma'_{Y'} + \Gamma''_{Y'}.$$

By taking a common resolution of $X \dashrightarrow Y'$, we see that $(Y', \Gamma_{Y'})$ is lc. Moreover we see that $\Gamma''_{Y'}$ is a reduced divisor. Indeed, we can write $g^* \Delta'' = g_*^{-1} \Delta'' + \sum a_i E_i$ where E_i are g -exceptional prime divisors and $a_i > 0$. Then $a(E_i, X, \Delta') > -1$ because $(X, \Delta = \Delta' + \Delta'')$ is lc. Therefore $\text{Supp}(\sum a_i E_i) \subset \text{Supp} F$. Since $F_{Y'} = 0$, we see that $\Gamma''_{Y'} = \phi_* g_*^{-1} \Delta''$ and thus $\Gamma''_{Y'}$ is a reduced divisor. Now we can easily check that $(Y', \Gamma_{Y'} = \Gamma'_{Y'} + \Gamma''_{Y'})$ satisfies all the conditions of Lemma 2.13. If we set $D' = f^* D$ and $E' = f^* E$, then $K_{Y'} + \Gamma_{Y'} \sim_{\mathbb{Q}} h'^* D'$ and $\Gamma''_{Y'} \sim_{\mathbb{Q}} h'^* E'$. By construction we can also check that $(Y', \Gamma_{Y'})$ is a log birational model of (X, Δ) . Therefore (X, Δ) has a weak lc model with semi-ample log canonical divisor if $(Y', \Gamma_{Y'})$ has a weak lc model with semi-ample log canonical divisor. In this way, by replacing $\pi : (X, \Delta) \rightarrow Z$ with $h' : (Y', \Gamma_{Y'}) \rightarrow \tilde{Z}$, we may assume the condition (i) of the lemma.

By the above discussions, we can assume the conditions (i), (ii) and (iii) of the lemma to prove Theorem 4.2. Finally we show that we can assume the condition (iv) of the lemma to prove Theorem 4.2. Since $K_X + \Delta$ is pseudo-effective, D is pseudo-effective (cf. [N, II 5.6 Lemma]). Let A be a general ample \mathbb{Q} -divisor on Z such that $A + E$ and $D + A$ are also ample. By Lemma 3.5, there is a sequence of birational maps of the D -MMP with scaling of A

$$(Z = Z_0, \lambda_0) \dashrightarrow \cdots \dashrightarrow (Z_i, \lambda_i) \dashrightarrow \cdots.$$

Then we have $\lim_{i \rightarrow \infty} \lambda_i = 0$ since D is pseudo-effective. Moreover, by construction of the D -MMP with scaling in Lemma 3.5, we see that $D_{Z_i} + \lambda_i A_{Z_i}$ is semi-ample if $\lambda_i > 0$.

Since A is a general ample \mathbb{Q} -divisor and $D + A$ is nef, by Lemma 3.6, we have the following diagram

$$\begin{array}{ccccccc} (X, \Delta) & \dashrightarrow & \cdots & \dashrightarrow & (X_{k_1}, \Delta_{X_{k_1}}) & \dashrightarrow & \cdots & \dashrightarrow & (X_{k_i}, \Delta_{X_{k_i}}) & \dashrightarrow & \cdots \\ \pi = \pi_0 \downarrow & & & & \pi_1 \downarrow & & & & \pi_i \downarrow & & \\ (Z_0, \lambda_0) & \dashrightarrow & \cdots & \dashrightarrow & (Z_1, \lambda_1) & \dashrightarrow & \cdots & \dashrightarrow & (Z_i, \lambda_i) & \dashrightarrow & \cdots \end{array}$$

such that

- (i) for any i , π_i is projective and surjective with connected fibers,
- (ii) the upper horizontal sequence of birational maps is a sequence of steps of the $(K_X + \Delta)$ -MMP with scaling of $\pi^* A$ such that if we set $k_0 = 0$ and

$$\lambda'_j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_j} + \Delta_{X_j} + \mu(\pi^* A)_{X_j} \text{ is nef}\},$$

then $\lambda'_j = \lambda_i$ for $k_i \leq j < k_{i+1}$, and

- (iii) for any two indices $i < i'$ and \mathbb{Q} -divisor B on Z_i , we have $(\pi_i^* B)_{X_{k_{i'}}$ = $\pi_{i'}^* B_{Z_{i'}}$.

Set $A' = \pi^* A$. Since $(X, 0)$ is klt and X is \mathbb{Q} -factorial, $(X_{k_i}, 0)$ is klt and X_{k_i} is \mathbb{Q} -factorial. Moreover we have $K_{X_{k_i}} + \Delta_{X_{k_i}} \sim_{\mathbb{Q}} \pi_i^* D_{Z_i}$ and $\Delta''_{X_{k_i}} \sim_{\mathbb{Q}} \pi_i^* E_{Z_i}$ by (iii) of the above properties. Taking a common resolution of $X \dashrightarrow X_{k_i}$, we see that $(X_{k_i}, \Delta_{X_{k_i}} = \Delta'_{X_{k_i}} + \Delta''_{X_{k_i}})$ satisfies all the conditions of Lemma 2.13 with respect to π_i . In this way we can replace $\pi : (X, \Delta) \rightarrow Z$ with $\pi_i : (X_{k_i}, \Delta_{X_{k_i}}) \rightarrow Z_i$ for some $i \gg 0$. We note that $K_{X_{k_i}} + \Delta_{X_{k_i}} + \lambda_i A'_{X_{k_i}} \sim_{\mathbb{Q}} \pi_i^* (D_{Z_i} + \lambda_i A_{Z_i})$ is semi-ample if $\lambda_i > 0$. In particular $K_{X_{k_i}} + \Delta_{X_{k_i}} + \lambda_i A'_{X_{k_i}}$ is movable if $\lambda_i > 0$.

If $\lambda_i = 0$ for some i , we may assume that $K_X + \Delta$ is nef by replacing $\pi : (X, \Delta) \rightarrow Z$ with $\pi_i : (X_{k_i}, \Delta_{X_{k_i}}) \rightarrow Z_i$. Then any ample \mathbb{Q} -divisor satisfies the condition (iv) of the lemma. If $\lambda_i > 0$ for any i , we replace $\pi : (X, \Delta) \rightarrow Z$ with $\pi_i : (X_{k_i}, \Delta_{X_{k_i}}) \rightarrow Z_i$ for some $i \gg 0$. Then

we can assume that the $(K_X + \Delta)$ -MMP contains only flips. Moreover $K_{X_{k_i}} + \Delta_{X_{k_i}} + \lambda_i A'_{X_{k_i}}$ is movable for any i . Therefore we see that $K_X + \Delta + \lambda_i A'$ is movable for any i . Since $\lim_{i \rightarrow \infty} \lambda_i = 0$, $K_X + \Delta + \delta A'$ is movable for any sufficiently small $\delta > 0$. In this way we can assume the condition (iv) of the lemma. So we are done. \square

5. PROOF OF THE MAIN RESULT

In this section we prove Theorem 4.2. Theorem 1.2 follows from Lemma 4.1 and Theorem 4.2 as we mentioned in Section 4. First we prove Theorem 4.2 assuming Proposition 5.1, Proposition 5.3 and Proposition 5.4. After that, we prove Proposition 5.1, Proposition 5.3 and Proposition 5.4.

Proof of Theorem 4.2. We prove it by the induction on d_0 . Clearly Theorem 4.2 holds when $d_0 = 0$. Pick any integer $d_0 > 0$ and assume Theorem 4.2 for $d_0 - 1$, and pick any $\pi : (X, \Delta) \rightarrow Z$ as in Theorem 4.2. By Lemma 4.5, we can assume that $\pi : (X, \Delta) \rightarrow Z$ satisfies all the conditions of Lemma 4.5. Moreover we may assume that $E \neq 0$ because otherwise Theorem 4.2 follows from Proposition 4.3 and the condition (iii) of Lemma 2.13. Then we have the following three cases.

Case 1. D is not big and $D - eE$ is pseudo-effective for a sufficiently small positive rational number e .

Case 2. D is not big and $D - eE$ is not pseudo-effective for any positive rational number e .

Case 3. D is big.

But then Theorem 4.2 follows from Proposition 5.1, Proposition 5.3 and Proposition 5.4 below. So we are done. \square

Next we prove Proposition 5.1, Proposition 5.3 and Proposition 5.4. From now on we freely use the notations of the conditions (ii) and (iii) of Lemma 4.5. We write the notations for the reader's convenience.

- $\Delta = \Delta' + \Delta''$ such that Δ' and Δ'' satisfy all the conditions of Lemma 2.13 with respect to π , and
- D is a pseudo-effective \mathbb{Q} -divisor and E is an effective \mathbb{Q} -divisor satisfying $K_X + \Delta \sim_{\mathbb{Q}} \pi^* D$ and $\Delta'' \sim_{\mathbb{Q}} \pi^* E$ respectively.

Proposition 5.1. *Fix a positive integer d_0 . Assume Theorem 4.2 for $d_0 - 1$, and assume existence of a good minimal model or a Mori fiber space for all d -dimensional projective Kawamata log terminal pairs with boundary \mathbb{Q} -divisors such that $d \leq d_0$.*

Let $\pi : (X, \Delta) \rightarrow Z$ be as in Theorem 4.2 satisfying all conditions of Lemma 4.5. Let D and E be \mathbb{Q} -divisors as in the condition (iii) of Lemma 4.5. Suppose that $E \neq 0$.

If D is not big and $D - eE$ is pseudo-effective for a sufficiently small positive rational number e , then (X, Δ) has a good minimal model.

Proof. We prove it with several steps.

Step 1. In this step we construct a diagram used in the proof.

Fix a strictly decreasing infinite sequence of rational numbers $\{a_n\}_{n \geq 1}$ such that $0 < a_n < e$ for any n and $\lim_{n \rightarrow \infty} a_n = 0$. Then $D - a_n E$ is pseudo-effective for all $n \geq 1$. By [FG1, Corollary 3.2] and the definition of D and E , there are \mathbb{Q} -divisors Ψ_n on Z such that all (Z, Ψ_n) are klt and $D - a_n E \sim_{\mathbb{Q}} K_Z + \Psi_n$. Then $K_X + \Delta - a_n \Delta'' \sim_{\mathbb{Q}} \pi^*(K_Z + \Psi_n)$.

By the hypothesis, (Z, Ψ_n) has a good minimal model. By running the $(K_Z + \Psi_n)$ -MMP with scaling of a general ample divisor, we get a good minimal model $(Z, \Psi_n) \dashrightarrow (Z', (\Psi_n)_{Z'})$ of (Z, Ψ_n) by [B2, Theorem 4.1 (iii)]. Set $Z_n = Z'$. By Lemma 3.6, we obtain the following diagram

$$\begin{array}{ccc} (X, \Delta - a_n \Delta'') & \xrightarrow{\phi_n} & (X_n, \Delta_{X_n} - a_n \Delta''_{X_n}) \\ \pi \downarrow & & \downarrow \pi_n \\ Z & \dashrightarrow & Z' = Z_n \end{array}$$

such that ϕ_n is a finitely many steps of the $(K_X + \Delta - a_n \Delta'')$ -MMP with scaling. Then

$$K_{X_n} + \Delta_{X_n} - a_n \Delta''_{X_n} \sim_{\mathbb{Q}} \pi_n^*(D_{Z_n} - a_n E_{Z_n}) \sim_{\mathbb{Q}} \pi_n^*(K_{Z_n} + (\Psi_n)_{Z_n})$$

by the condition (iii) of Lemma 3.6. Therefore $K_{X_n} + \Delta_{X_n} - a_n \Delta''_{X_n}$ is semi-ample and thus $(X_n, \Delta_{X_n} - a_n \Delta''_{X_n})$ is a good minimal model of $(X, \Delta - a_n \Delta'')$. By construction we can easily check that $(X_n, 0)$ is klt. We note that $\pi_n : X_n \rightarrow Z_n$ is projective and surjective with connected fibers by the condition (i) of Lemma 3.6. We also note that $K_{X_n} + \Delta_{X_n} - t \Delta''_{X_n} \sim_{\mathbb{Q}} \pi_n^*(D_{Z_n} - t E_{Z_n})$ for any $t \geq 0$ by the condition (iii) of Lemma 3.6.

Step 2. In this step we prove that there are infinitely many indices n such that (X_n, Δ_{X_n}) is lc, where X_n was constructed in Step 1 by running the $(K_X + \Delta - a_n \Delta'')$ -MMP.

Suppose by contradiction that there are only finitely many indices n such that (X_n, Δ_{X_n}) is lc. Fix n_0 such that (X_i, Δ_{X_i}) is not lc for every $i \geq n_0$. Consider

$$I = \{M \in \mathbb{R}_{\geq 0} \mid M = \text{lct}(X_i, \Delta'_{X_i}; \Delta''_{X_i}), i \geq n_0\}$$

where Δ'_{X_i} (resp. Δ''_{X_i}) is the birational transform of Δ' (resp. Δ'') on X_i . Then I does not contain one by our assumption. On the other hand, since Δ'_{X_i} is the birational transform of Δ' on X_i , any coefficient of component in Δ'_{X_i} is in a finite set which does not depend on i . Moreover Δ''_{X_i} is a reduced divisor and $\text{lct}(X_i, \Delta'_{X_i}; \Delta''_{X_i}) \geq 1 - a_i$ by construction. Since $\lim_{n \rightarrow \infty} a_n = 0$, by the ACC for log canonical thresholds (cf. Theorem 2.10), the set I must contain one. In this way we get a contradiction and thus there are infinitely many indices n such that (X_n, Δ_{X_n}) is lc.

Step 3. By taking a common resolution of ϕ_n , we can check that $(X_n, \Delta_{X_n} = \Delta'_{X_n} + \Delta''_{X_n})$ satisfies all the conditions of Lemma 2.13 with respect to π_n . By Step 2 and replacing $\{a_n\}_{n \geq 1}$ with its subsequence, we get a strictly decreasing infinite sequence $\{a_n\}_{n \geq 1}$ of positive rational numbers such that

- (i) $a_n < e$ for any n and $\lim_{n \rightarrow \infty} a_n = 0$, and
- (ii) for any $n \geq 1$, there is a diagram

$$\begin{array}{ccc} (X, \Delta - a_n \Delta'') & \xrightarrow{\phi_n} & (X_n, \Delta_{X_n} - a_n \Delta''_{X_n}) \\ \pi \downarrow & & \downarrow \pi_n \\ Z & \dashrightarrow & Z_n \end{array}$$

such that

- (ii-a) X_n and Z_n are \mathbb{Q} -factorial, (X_n, Δ_{X_n}) is lc, $(X_n, 0)$ is klt and π_n is a projective surjective morphism with connected fibers,
- (ii-b) $(X_n, \Delta_{X_n} = \Delta'_{X_n} + \Delta''_{X_n})$ satisfies all the conditions of Lemma 2.13 with respect to π_n , $K_{X_n} + \Delta_{X_n} \sim_{\mathbb{Q}} \pi_n^* D_{Z_n}$ and $\Delta''_{X_n} \sim_{\mathbb{Q}} \pi_n^* E_{Z_n}$, and
- (ii-c) ϕ_n is a finitely many steps of the $(K_X + \Delta - a_n \Delta'')$ -MMP to a good minimal model $(X_n, \Delta_{X_n} - a_n \Delta''_{X_n})$.

By replacing $\{a_n\}_{n \geq 1}$ with its subsequence again, we also have that

- (ii-d) X_i and X_j are isomorphic in codimension one for any i and j .

Indeed, for any $n \geq 1$, every prime divisor contracted by ϕ_n is a component of $N_\sigma(K_X + \Delta - a_n \Delta'')$. Recall that e is a positive rational number such that $D - eE$ is pseudo-effective. Thus $K_X + \Delta - e\Delta''$ is also pseudo-effective. By the basic property of $N_\sigma(\cdot)$, we have

$$\begin{aligned} & N_\sigma(K_X + \Delta - a_n \Delta'') \\ & \leq \left(1 - \frac{a_n}{e}\right) N_\sigma(K_X + \Delta) + \frac{a_n}{e} N_\sigma(K_X + \Delta - e\Delta''). \end{aligned}$$

Thus every prime divisor contracted by ϕ_n is also a component of $N_\sigma(K_X + \Delta) + N_\sigma(K_X + \Delta - e\Delta'')$, which does not depend on n . Therefore, we can replace $\{a_n\}_{n \geq 1}$ with its subsequence so that X_i and X_j are isomorphic in codimension one for any i and j .

We note that $D_{Z_n} - (a_n - \delta)E_{Z_n}$ is not big for any sufficiently small rational number $\delta > 0$. Indeed, the birational map $Z \dashrightarrow Z_n$ is a finitely many steps of the $(D - a_n E)$ -MMP. Then $Z \dashrightarrow Z_n$ is also a finitely many steps of the $(D - (a_n - \delta)E)$ -MMP for any sufficiently small $\delta \geq 0$. Since $D - tE$ is not big for any $t \geq 0$, we see that $D_{Z_n} - (a_n - \delta)E_{Z_n}$ is not big for any sufficiently small $\delta > 0$.

Step 4. Suppose that (X_1, Δ_{X_1}) has a good minimal model. Then we can show that (X, Δ) has a weak lc model with semi-ample log canonical divisor. Indeed, by [B2, Theorem 4.1 (iii)] and by running the $(K_{X_1} + \Delta_{X_1})$ -MMP with scaling of an ample divisor, we get a finitely many steps of the $(K_{X_1} + \Delta_{X_1})$ -MMP to a good minimal model $\psi : (X_1, \Delta_{X_1}) \dashrightarrow (X', \Delta_{X'})$ of (X_1, Δ_{X_1}) . By construction $K_{X'} + \Delta_{X'}$ is semi-ample. Fix a sufficiently small positive rational number $t \ll a_1$. Then ψ is also a finitely many steps of the $(K_{X_1} + \Delta_{X_1} - t\Delta''_{X_1})$ -MMP. We note that any lc center of $(X_1, \Delta_{X_1} - t\Delta''_{X_1})$ dominates Z_1 . By Proposition 4.3, $(X_1, \Delta_{X_1} - t\Delta''_{X_1})$ has a good minimal model, and thus $(X', \Delta_{X'} - t\Delta''_{X'})$ has a good minimal model. Therefore we can run the $(K_{X'} + \Delta_{X'} - t\Delta''_{X'})$ -MMP with scaling of an ample divisor and obtain a good minimal model $\psi' : (X', \Delta_{X'} - t\Delta''_{X'}) \dashrightarrow (X'', \Delta_{X''} - t\Delta''_{X''})$. Now we get the following sequence of birational maps

$$X \xrightarrow{\phi_1} X_1 \xrightarrow{\psi} X' \xrightarrow{\psi'} X'',$$

where ϕ_1 (resp. ψ, ψ') is a sequence of steps the $(K_X + \Delta - a_1\Delta'')$ -MMP (resp. the $(K_{X_1} + \Delta_{X_1})$ -MMP, the $(K_{X'} + \Delta_{X'} - t\Delta''_{X'})$ -MMP) to a good minimal model. Since we pick $t > 0$ sufficiently small, by the standard argument of the length of extremal rays, we see that $K_{X''} + \Delta''_{X''}$ is also semi-ample (see, for example, the proof of [B1, Proposition 3.2 (5)] or the proof of [H, Theorem 1.2]).

We prove that X_1 and X'' are isomorphic in codimension one. More precisely, we prove that both ψ and ψ' contain only flips. We note that ϕ_1 is in particular a birational contraction. Recall that there is a \mathbb{Q} -divisor A on X such that $K_X + \Delta + \delta A$ is movable for any sufficiently small $\delta > 0$, which is the condition (iv) of Lemma 4.5. Therefore we see that $K_{X_1} + \Delta_{X_1} + \delta A_{X_1}$ is movable for any sufficiently small $\delta > 0$. Then $N_\sigma(K_{X_1} + \Delta_{X_1}) = 0$, and ψ contains only flips. Thus X_1 and X' are isomorphic in codimension one. By the condition (ii-c) of Step 3, $K_{X'} + \Delta_{X'} - a_1\Delta''_{X'}$ is movable. Then $N_\sigma(K_{X'} + \Delta_{X'} - a_1\Delta''_{X'}) = 0$.

Moreover $N_\sigma(K_{X'} + \Delta_{X'}) = 0$ because $K_{X'} + \Delta_{X'}$ is semi-ample. Since we choose t satisfying $0 < t < a_1$, we have

$$\begin{aligned} & N_\sigma(K_{X'} + \Delta_{X'} - t\Delta_{X'}'') \\ & \leq \left(1 - \frac{t}{a_1}\right) N_\sigma(K_{X'} + \Delta_{X'}) + \frac{t}{a_1} N_\sigma(K_{X'} + \Delta_{X'} - a_1\Delta_{X'}'') \\ & = 0 \end{aligned}$$

and ψ' contains only flips. Therefore X' and X'' are isomorphic in codimension one. Then X_1 and X'' are isomorphic in codimension one.

Recall that X_i and X_j are isomorphic in codimension one, which is the condition (ii-d) of Step 3. Therefore X_n and X'' are isomorphic in codimension one for any n . By the condition (i) in Step 3, we have $t \geq a_n$ for any $n \gg 0$, where $\{a_n\}_{n \geq 1}$ is defined in Step 3. Since $K_{X''} + \Delta_{X''}$ and $K_{X''} + \Delta_{X''} - t\Delta_{X''}''$ are semi-ample, $K_{X''} + \Delta_{X''} - a_n\Delta_{X''}''$ is also semi-ample for any $n \gg 0$. By the condition (ii-c) of Step 3, $(X_n, \Delta_{X_n} - a_n\Delta_{X_n}'')$ is a good minimal model of $(X, \Delta - a_n\Delta'')$. From these facts, we see that $(X'', \Delta_{X''} - a_n\Delta_{X''}'')$ is also a good minimal model of $(X, \Delta - a_n\Delta'')$ for any $n \gg 0$. Let $p : Y \rightarrow X$ and $q : Y \rightarrow X''$ be a common resolution of $X \dashrightarrow X''$. Then

$$p^*(K_X + \Delta - a_n\Delta'') - q^*(K_{X''} + \Delta_{X''} - a_n\Delta_{X''}'') \geq 0$$

for any $n \gg 0$. Since $\lim_{n \rightarrow \infty} a_n = 0$, which is the condition (i) in Step 3, we have

$$p^*(K_X + \Delta) - q^*(K_{X''} + \Delta_{X''}) \geq 0$$

by considering the limit. Since $K_{X''} + \Delta_{X''}$ is semi-ample, we see that $(X'', \Delta_{X''})$ is a weak lc model of (X, Δ) with semi-ample log canonical divisor.

By the above arguments, to prove Proposition 5.1, we only have to prove that (X_1, Δ_{X_1}) has a good minimal model.

Step 5. Finally we prove that (X_1, Δ_{X_1}) has a good minimal model. If $E_{Z_1} = 0$, then $K_{X_1} + \Delta_{X_1} = K_{X_1} + \Delta_{X_1} - a_1\Delta_{X_1}''$ is semi-ample. Therefore we may assume that $E_{Z_1} \neq 0$. Recall again that any lc center of $(X_1, \Delta_{X_1} - t\Delta_{X_1}'')$ dominates Z_1 for any $0 < t \leq a_1$.

Pick a sufficiently small positive rational number $u \ll a_1$. By [FG1, Corollary 3.2], there is a \mathbb{Q} -divisor Ψ' on Z_1 such that (Z_1, Ψ') is klt and $D_{Z_1} - (a_1 - u)E_{Z_1} \sim_{\mathbb{Q}} K_{Z_1} + \Psi'$. Then (Z_1, Ψ') has a good minimal model by the hypothesis. Run the $(K_{Z_1} + \Psi')$ -MMP with scaling of an ample divisor. By [B2, Theorem 4.1 (iii)], we obtain a good minimal model $(Z_1, \Psi') \dashrightarrow (Z_1', \Psi'_{Z_1'})$ of (Z_1, Ψ') . Furthermore, by Lemma 3.6,

we obtain the following diagram

$$\begin{array}{ccc}
 (X_1, \Delta_{X_1} - (a_1 - u)\Delta''_{X_1}) & \xrightarrow{\phi} & (X'_1, \Delta_{X'_1} - (a_1 - u)\Delta''_{X'_1}) \\
 \pi_1 \downarrow & & \pi'_1 \downarrow \\
 Z_1 & \dashrightarrow & Z'_1
 \end{array}$$

such that the upper horizontal birational map ϕ is a finitely many steps of the $(K_{X_1} + \Delta_{X_1} - (a_1 - u)\Delta''_{X_1})$ -MMP to a good minimal model. Since $K_{X_1} + \Delta_{X_1} - a_1\Delta''_{X_1}$ is semi-ample and $0 < u \ll a_1$ is sufficiently small, by the standard argument of the length of extremal rays, we see that ϕ is also a finitely many steps of the $(K_{X_1} + \Delta_{X_1})$ -MMP and $K_{X'_1} + \Delta_{X'_1} - a_1\Delta''_{X'_1}$ is semi-ample (cf. [B1, Proposition 3.2 (5)]). From this we also see that $(X'_1, \Delta_{X'_1})$ is lc. Since we have

$$\begin{aligned}
 K_{X'_1} + \Delta_{X'_1} - (a_1 - u)\Delta''_{X'_1} &\sim_{\mathbb{Q}} \pi_1'^*(D_{Z'_1} - (a_1 - u)E_{Z'_1}) \quad \text{and} \\
 K_{X'_1} + \Delta_{X'_1} - a_1\Delta''_{X'_1} &\sim_{\mathbb{Q}} \pi_1'^*(D_{Z'_1} - a_1E_{Z'_1}),
 \end{aligned}$$

$D_{Z'_1} - (a_1 - u)E_{Z'_1}$ and $D_{Z'_1} - a_1E_{Z'_1}$ are semi-ample.

Recall that $D_{Z_1} - (a_1 - \delta)E_{Z_1}$ is not big for any sufficiently small rational number $\delta > 0$. Therefore $D_{Z'_1} - (a_1 - \delta)E_{Z'_1}$ is not big for any sufficiently small rational number $\delta > 0$. Pick two sufficiently small rational numbers u_1 and u_2 satisfying $0 < u_1 < u_2 < u$. Then we see that $D_{Z'_1} - (a_1 - u_i)E_{Z'_1}$ is semi-ample for $i = 1, 2$ because these are represented by a $\mathbb{Q}_{>0}$ -linear combination of $D_{Z'_1} - (a_1 - u)E_{Z'_1}$ and $D_{Z'_1} - a_1E_{Z'_1}$. Moreover $D_{Z'_1} - (a_1 - u_i)E_{Z'_1}$ is not big because u_1 and u_2 are sufficiently small. For $i = 1, 2$, let $f_i : Z'_1 \rightarrow W_i$ be the Stein factorization of the projective morphism induced by $D_{Z'_1} - (a_1 - u_i)E_{Z'_1}$. Then $W_1 \simeq W_2$. Indeed, let C be a curve on Z'_1 . Then

$$\begin{aligned}
 &C \text{ is contracted by } f_1 \\
 &\Leftrightarrow C \cdot (D_{Z'_1} - (a_1 - u_1)E_{Z'_1}) = 0 \\
 &\Leftrightarrow C \cdot (D_{Z'_1} - a_1E_{Z'_1}) = C \cdot (D_{Z'_1} - (a_1 - u)E_{Z'_1}) = 0 \\
 &\Leftrightarrow C \cdot (D_{Z'_1} - (a_1 - u_2)E_{Z'_1}) = 0 \\
 &\Leftrightarrow C \text{ is contracted by } f_2.
 \end{aligned}$$

Thus $W_1 \simeq W_2$. Set $f : Z'_1 \rightarrow W = W_1 \simeq W_2$. By construction, we see that $D_{Z'_1} - (a_1 - u_i)E_{Z'_1}$ is trivial over W for $i = 1, 2$. Then $E_{Z'_1} \sim_{\mathbb{Q}, W} 0$, and moreover $D_{Z'_1} \sim_{\mathbb{Q}, W} 0$. Therefore $K_{X'_1} + \Delta_{X'_1} \sim_{\mathbb{Q}, W} 0$. We also have $\dim W < \dim Z'_1$ because $D_{Z'_1} - (a_1 - u_1)E_{Z'_1}$ is not big. Since we assume Theorem 4.2 for $d_0 - 1$, by applying this hypothesis to $f \circ \pi'_1 : (X'_1, \Delta_{X'_1}) \rightarrow W$, we see that $(X'_1, \Delta_{X'_1})$ has a good minimal

model. Then (X_1, Δ_{X_1}) also has a good minimal model by construction. Thus we complete the proof. \square

Remark 5.2. In the proof of Proposition 5.1, we use the condition that $D - eE$ is pseudo-effective from the start. On the other hand, we do not use the condition that D is not big until the final part of Step 5. Therefore we can use the same discussions as in Step 1, Step 2, Step 3, Step 4 and the first half of Step 5 to prove Theorem 4.2 in Case 3.

Proposition 5.3. *Fix a positive integer d_0 . Assume Theorem 4.2 for $d_0 - 1$, and assume existence of a good minimal model or a Mori fiber space for all d -dimensional projective Kawamata log terminal pairs with boundary \mathbb{Q} -divisors such that $d \leq d_0$.*

Let $\pi : (X, \Delta) \rightarrow Z$ be as in Theorem 4.2 satisfying all conditions of Lemma 4.5. Let D and E be \mathbb{Q} -divisors as in the condition (iii) of Lemma 4.5. Suppose that $E \neq 0$.

If D is not big and $D - eE$ is not pseudo-effective for any positive rational number e , then (X, Δ) has a good minimal model.

Proof. We prove it by using similar techniques used in the proof of Proposition 5.1.

Step 1. In this step we construct a diagram used in the proof.

Let $\{\epsilon_n\}_{n \geq 1}$ be a strictly decreasing infinite sequence of rational numbers such that $0 < \epsilon_n < 1$ for any n and $\lim_{n \rightarrow \infty} \epsilon_n = 0$. By [FG1, Corollary 3.2] and the definition of D and E , there are \mathbb{Q} -divisors Ψ_n on Z such that all (Z, Ψ_n) are klt and $D - \epsilon_n E \sim_{\mathbb{Q}} K_Z + \Psi_n$. Fix a sufficiently general ample \mathbb{Q} -divisor A on Z such that $D + (1/2)A$ and $(1/2)A - E$ are nef, $(X, \Delta + \pi^*A)$ is lc and $(Z, \Psi_n + A)$ is klt for any n . Then

$$\begin{aligned} K_Z + \Psi_n + A &\sim_{\mathbb{Q}} D - \epsilon_n E + A \\ &= D + \frac{1}{2}A + \epsilon_n \left(\frac{1}{2}A - E \right) + \frac{1}{2}(1 - \epsilon_n)A \end{aligned}$$

is nef. Therefore we can run the $(K_Z + \Psi_n)$ -MMP with scaling of A , and we get a Mori fiber space

$$(Z, \Psi_n) \dashrightarrow (Z', (\Psi_n)_{Z'}) \xrightarrow{f_n} W_n.$$

Set $Z_n = Z'$. Let $\tau_n = \tau(Z, \Psi_n; A)$ be the pseudo-effective threshold of A with respect to (Z, Ψ_n) . Then $0 < \tau_n \leq 1$ since $K_Z + \Psi_n$ is not pseudo-effective and $K_Z + \Psi_n + A$ is nef. By the basic properties of the log MMP with scaling, $D_{Z_n} - \epsilon_n E_{Z_n} + \tau_n A_{Z_n} \sim_{\mathbb{Q}} K_{Z_n} + (\Psi_n)_{Z_n} + \tau_n A_{Z_n}$ is nef and trivial over W_n . Clearly $D_{Z_n} - \epsilon_n E_{Z_n} \sim_{\mathbb{Q}} K_{Z_n} + (\Psi_n)_{Z_n}$ is

anti-ample over W_n by construction. On the other hand, D_{Z_n} is nef over W_n because it is pseudo-effective. Therefore E_{Z_n} is ample over W_n . By Lemma 3.6, we obtain the following diagram

$$\begin{array}{ccc} (X, \Delta - \epsilon_n \Delta'') & \dashrightarrow & (X_n, \Delta_{X_n} - \epsilon_n \Delta''_{X_n}) \\ \pi \downarrow & & \downarrow \pi_n \\ Z & \dashrightarrow & Z_n \xrightarrow{f_n} W_n \end{array}$$

such that

- (i) the upper horizontal birational map is a finitely many steps of the $(K_X + \Delta - \epsilon_n \Delta'')$ -MMP,
- (ii) π_n is projective and surjective with connected fibers, and
- (iii) $K_{X_n} + \Delta_{X_n} \sim_{\mathbb{Q}} \pi_n^* D_{Z_n}$ and $\Delta''_{X_n} \sim_{\mathbb{Q}} \pi_n^* E_{Z_n}$ for any n .

Step 2. In this step we prove that there is an index n such that (X_n, Δ_{X_n}) is lc and D_{Z_n} is trivial over W_n . The idea is similar to the proof of [DHP, Proposition 8.7] or [G, Lemma 3.1]. By the cone theorem [F4, Theorem 4.5.2], D_{Z_n} is trivial over W_n if and only if D_{Z_n} is numerically trivial over W_n .

By the same arguments as in Step 2 in the proof of Proposition 5.1, we can find infinitely many indices n such that (X_n, Δ_{X_n}) is lc. Therefore, by replacing ϵ_n with its subsequence, we may assume that (X_n, Δ_{X_n}) is lc for any n . Moreover we may assume that the dimension of W_n is constant for all n by replacing ϵ_n with its subsequence.

Recall that D_{Z_n} is nef over W_n . On the other hand, $D_{Z_n} - \epsilon_n E_{Z_n}$ is anti-ample over W_n . Therefore $D_{Z_n} - \nu_n E_{Z_n}$ is numerically trivial over W_n for some ν_n and we have $0 \leq \nu_n < \epsilon_n$. Then we have

$$\begin{aligned} K_{X_n} + \Delta_{X_n} - \nu_n \Delta''_{X_n} \\ = K_{X_n} + \Delta'_{X_n} + (1 - \nu_n) \Delta''_{X_n} \sim_{\mathbb{Q}} \pi_n^* (D_{Z_n} - \nu_n E_{Z_n}) \equiv_{W_n} 0. \end{aligned}$$

Let F_n be the general fiber of $f_n \circ \pi_n$. Then $(F_n, (\Delta_{X_n} - \nu_n \Delta''_{X_n})|_{F_n})$ is lc, and $\Delta''_{X_n}|_{F_n} \sim_{\mathbb{Q}} (\pi_n^* E)|_{F_n}$ is not numerically trivial since E_{Z_n} is ample over W_n . Consider

$$T = \{\nu \in \mathbb{R}_{\geq 0} \mid K_{F_n} + \Delta'_{X_n}|_{F_n} + \nu \Delta''_{X_n}|_{F_n} \equiv 0 \text{ for some } n\}.$$

Clearly $T \supset \{1 - \nu_n\}_{n \geq 1}$ by the definition of ν_n . Conversely, $\nu = 1 - \nu_n$ is the unique number satisfying $K_{F_n} + \Delta'_{X_n}|_{F_n} + \nu \Delta''_{X_n}|_{F_n} \equiv 0$ because $\Delta''_{X_n}|_{F_n}$ is not numerically trivial. Therefore we have $T = \{1 - \nu_n\}_{n \geq 1}$. By construction, the dimension of F_n is constant for any n and any coefficient of component of $\Delta'_{X_n}|_{F_n}$ or $\Delta''_{X_n}|_{F_n}$ is in a finite set which does not depend on n . Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and $0 \leq \nu_n < \epsilon_n$, T must contain one by the ACC for numerically trivial pairs (cf. Theorem 2.11).

Then $\nu_n = 0$ for some n . In this way, we see that there is an index n such that (X_n, Δ_{X_n}) is lc and D_{Z_n} is trivial over W_n .

Step 3. Fix an index n such that (X_n, Δ_{X_n}) is lc and D_{Z_n} is trivial over W_n . Recall that $K_Z + \Psi_n \sim_{\mathbb{Q}} D - \epsilon_n E$ and τ_n is the pseudo-effective threshold of A with respect to (Z, Ψ_n) . In this step we prove that $D - t(\epsilon_n E - \tau_n A)$ is not big for any rational number $0 \leq t \leq 1$. Note that $D - t(\epsilon_n E - \tau_n A)$ is pseudo-effective for any $0 \leq t \leq 1$ because it is represented by a $\mathbb{Q}_{\geq 0}$ -linear combination of D and $D - \epsilon_n E + \tau_n A$.

Suppose by contradiction that $D - t(\epsilon_n E - \tau_n A)$ is big for some $0 \leq t \leq 1$. Then $D_{Z_n} - t(\epsilon_n E_{Z_n} - \tau_n A_{Z_n})$ is also big. On the other hand, D_{Z_n} is trivial over W_n . Moreover $D_{Z_n} - \epsilon_n E_{Z_n} + \tau_n A_{Z_n}$ is also trivial over W_n as we mentioned in Step 1. Therefore $D_{Z_n} - t(\epsilon_n E_{Z_n} - \tau_n A_{Z_n})$ is trivial over W_n and thus it is \mathbb{Q} -linearly equivalent to the pullback of a \mathbb{Q} -divisor on W_n . Because $\dim W_n < \dim Z_n$, we get a contradiction. Therefore $D - t(\epsilon_n E - \tau_n A)$ is not big for any rational number $0 \leq t \leq 1$.

For this n , we put $\epsilon = \epsilon_n$ and $\tau = \tau_n$ in the rest of the proof. Since $K_X + \Delta - t(\epsilon \Delta'' - \tau \pi^* A) \sim_{\mathbb{Q}} \pi^*(D - t(\epsilon E - \tau A))$, $K_X + \Delta - t(\epsilon \Delta'' - \tau \pi^* A)$ is also pseudo-effective for any $0 \leq t \leq 1$.

Step 4. We set

$$\tilde{E} = E - \frac{\tau}{\epsilon} A \quad \text{and} \quad \tilde{\Delta} = \Delta'' - \frac{\tau}{\epsilon} \pi^* A \sim_{\mathbb{Q}} \pi^* \tilde{E}.$$

Note that \tilde{E} and $\tilde{\Delta}$ may not be effective. We see that $D - t\tilde{E}$ is pseudo-effective for any $0 \leq t \leq \epsilon$ because $D - t\tilde{E} = D - (t/\epsilon)(\epsilon E - \tau A)$. Since $K_X + \Delta - t\tilde{\Delta} \sim_{\mathbb{Q}} \pi^*(D - t\tilde{E})$, $K_X + \Delta - t\tilde{\Delta}$ is also pseudo-effective for any $0 \leq t \leq \epsilon$. Moreover, for any $0 \leq t \leq \epsilon$, $(X, \Delta - t\tilde{\Delta})$ is lc. To see this, recall that $0 < \tau \leq 1$, which is mentioned in Step 1, and that we picked A in Step 1 satisfying that $(X, \Delta + \pi^* A)$ is lc. Then we have $0 \leq t\tau/\epsilon \leq 1$ for any $0 \leq t \leq \epsilon$. Since $\Delta - t\tilde{\Delta} = \Delta - t\Delta'' + (t\tau/\epsilon)\pi^* A$, $(X, \Delta - t\tilde{\Delta})$ is indeed lc for any $0 \leq t \leq \epsilon$.

Step 5. From this step we use the same arguments as in the proof of Proposition 5.1. We only write down the outline of the proof.

Fix a strictly decreasing infinite sequence of rational numbers $\{a_n\}_{n \geq 1}$ such that $0 < a_n < \epsilon$ for any n and $\lim_{n \rightarrow \infty} a_n = 0$. With the divisors D and \tilde{E} , we carry out the same arguments as in Step 1 in the proof of Proposition 5.1. When we apply the arguments of Step 2 in the proof of Proposition 5.1, a minor change is needed. More precisely, we need to carry out the arguments with the effective part of $\tilde{\Delta}$. But we can eventually obtain the same result, that is, (X_n, Δ_{X_n}) is lc for infinitely many indices n . By the same arguments as in Step 3 in the

proof of Proposition 5.1, replacing $\{a_n\}_{n \geq 1}$ with its subsequence, we get a strictly decreasing infinite sequence $\{a_n\}_{n \geq 1}$ of positive rational numbers such that

- (i) $a_n < \epsilon$ for any n and $\lim_{n \rightarrow \infty} a_n = 0$, and
- (ii) for any $n \geq 1$, there is a diagram

$$\begin{array}{ccc} (X, \Delta - a_n \tilde{\Delta}) & \xrightarrow{\phi_n} & (X_n, \Delta_{X_n} - a_n \tilde{\Delta}_{X_n}) \\ \pi \downarrow & & \downarrow \pi_n \\ Z & \dashrightarrow & Z_n \end{array}$$

such that

- (ii-a) X_n and Z_n are \mathbb{Q} -factorial, (X_n, Δ_{X_n}) is lc, $(X_n, 0)$ is klt and π_n is a projective surjective morphism with connected fibers,
- (ii-b) $K_{X_n} + \Delta_{X_n} \sim_{\mathbb{Q}} \pi_n^* D_{Z_n}$ and $\tilde{\Delta}_{X_n} \sim_{\mathbb{Q}} \pi_n^* \tilde{E}_{Z_n}$, and
- (ii-c) ϕ_n is a finitely many steps of the $(K_X + \Delta - a_n \tilde{\Delta})$ -MMP to a good minimal model $(X_n, \Delta_{X_n} - a_n \tilde{\Delta}_{X_n})$.

By replacing $\{a_n\}_{n \geq 1}$ with its subsequence again, we also have that

- (ii-d) X_i and X_j are isomorphic in codimension one for any i and j .

We only check the condition (ii-d) because the others are trivial by construction. We note that any prime divisor contracted by ϕ_n is a component of $N_\sigma(K_X + \Delta - a_n \tilde{\Delta})$. Recall that ϵ is a positive rational number such that $D - t\tilde{E}$ is pseudo-effective for any $0 \leq t \leq \epsilon$, and that $K_X + \Delta - \epsilon \tilde{\Delta}$ is also pseudo-effective. By the basic property of $N_\sigma(\cdot)$, we have

$$\begin{aligned} & N_\sigma(K_X + \Delta - a_n \tilde{\Delta}) \\ & \leq \left(1 - \frac{a_n}{\epsilon}\right) N_\sigma(K_X + \Delta) + \frac{a_n}{\epsilon} N_\sigma(K_X + \Delta - \epsilon \tilde{\Delta}). \end{aligned}$$

Therefore any prime divisor contracted by ϕ_n is also a component of $N_\sigma(K_X + \Delta) + N_\sigma(K_X + \Delta - \epsilon \tilde{\Delta})$, which does not depend on n . In this way, we can replace $\{a_n\}_{n \geq 1}$ with its subsequence to satisfy that X_i and X_j are isomorphic in codimension one for any i and j .

We note that $D_{Z_n} - (a_n - \delta)\tilde{E}_{Z_n}$ is not big for any sufficiently small rational number $\delta > 0$. Indeed, the birational map $Z \dashrightarrow Z_n$ is a finitely many steps of the $(D - a_n \tilde{E})$ -MMP. Then $Z \dashrightarrow Z_n$ is also a finitely many steps of the $(D - (a_n - \delta)\tilde{E})$ -MMP for any sufficiently small $\delta > 0$. By the definition of \tilde{E} and by Step 3, $D - t\tilde{E}$ is not big

for any $0 \leq t \leq \epsilon$. Therefore we see that $D_{Z_n} - (a_n - \delta)\tilde{E}_{Z_n}$ is not big for any sufficiently small $\delta > 0$.

Step 6. Finally we complete the proof by using the same arguments as in Step 4 and Step 5 in the proof of Proposition 5.1. To carry out, we only check that every lc center of $(X_1, \Delta_{X_1} - t\tilde{\Delta}_{X_1})$ dominates Z_1 for any $0 < t \leq a_1$. Once we can check this, we see that (X_1, Δ_{X_1}) has a good minimal model by the same arguments as in Step 5 in the proof of Proposition 5.1, and thus (X, Δ) has a weak lc model with semi-ample log canonical divisor by the same arguments as in Step 4 in the proof of Proposition 5.1.

For any $0 < t \leq a_1$, every lc center of $(X_1, \Delta_{X_1} - t\tilde{\Delta}_{X_1})$ is also an lc center of $(X_1, \Delta_{X_1} - a_1\tilde{\Delta}_{X_1})$ because (X_1, Δ_{X_1}) is lc. Therefore we may check the condition only when $t = a_1$.

Recall again that τ is a rational number such that $0 < \tau \leq 1$. Since $a_1 < \epsilon$, we have $a_1\tau/\epsilon < 1$. Since $(X, \Delta + \pi^*A)$ is lc and

$$\Delta - a_1\tilde{\Delta} = \Delta - a_1\Delta'' + (a_1\tau/\epsilon)\pi^*A \leq \Delta + \pi^*A,$$

every lc center of $(X, \Delta - a_1\tilde{\Delta})$ is an lc center of $(X, \Delta + \pi^*A)$, and moreover it is also an lc center of $(X, \Delta - \Delta'')$. Since any lc center of $(X, \Delta - \Delta'')$ dominates Z by the condition (iii) of Lemma 2.13, we see that any lc center of $(X, \Delta - a_1\tilde{\Delta})$ dominates Z . Since ϕ is a finitely many steps of the $(K_X + \Delta - a_1\tilde{\Delta})$ -MMP, any lc center of $(X_1, \Delta_{X_1} - a_1\tilde{\Delta}_{X_1})$ dominates Z_1 . Thus we complete the proof. \square

Finally we prove Theorem 4.2 in Case 3. As we state in Proposition 5.4 below, we can in fact prove the case with assumptions weaker than Proposition 5.1 or Proposition 5.3.

Proposition 5.4. *Fix a positive integer d_0 . Assume Theorem 4.2 for $d_0 - 1$, and assume existence of a good minimal model or a Mori fiber space for all d -dimensional projective Kawamata log terminal pairs with boundary \mathbb{Q} -divisors such that $d \leq d_0 - 1$.*

*Let $\pi : (X, \Delta) \rightarrow Z$ be as in Theorem 4.2 satisfying all conditions of Lemma 4.5. Let D be as in the condition (iii) of Lemma 4.5, that is, a \mathbb{Q} -divisor on Z such that $K_X + \Delta \sim_{\mathbb{Q}} \pi^*D$.*

If D is big, then (X, Δ) has a good minimal model.

Proof. Let E be as in the condition (iii) of Lemma 4.5, that is, an effective \mathbb{Q} -divisor such that $\Delta'' \sim_{\mathbb{Q}} \pi^*E$. We may assume that $E \neq 0$ because otherwise the proposition follows from Proposition 4.4. Fix a

sufficiently small positive rational number $e < 1$ such that $D - eE$ is big. We prove the proposition with several steps.

Step 1. Compared with Proposition 5.1, we do not assume that

- (*) there is a good minimal model or a Mori fiber space for all projective klt pairs of dimension d_0 with boundary \mathbb{Q} -divisors.

If we assume the condition (*), we can apply the proof of Proposition 5.1 by Remark 5.2. Even though we do not assume the condition (*), we can apply the proof of Proposition 5.1 with only minor changes. In this step we check this.

Fix a strictly decreasing infinite sequence of rational numbers $\{a_n\}_{n \geq 1}$ such that $0 < a_n < e$ for any n and $\lim_{n \rightarrow \infty} a_n = 0$. In the proof of Proposition 5.1, the assumption (*) is used in Step 1, Step 4 and Step 5. In Step 1, we use the assumption (*) only to construct a sequence of steps of the $(K_Z + \Psi_n)$ -MMP with scaling to a good minimal model $(Z, \Psi_n) \dashrightarrow (Z_n, (\Psi_n)_{Z_n})$, where (Z, Ψ_n) is klt and $K_Z + \Psi_n \sim_{\mathbb{Q}} D - a_n E$. In our situation, we can construct this log MMP without the assumption (*) because $D - eE$ is big and we can apply [BCHM, Corollary 1.4.2]. Therefore we can carry out the same arguments as in Step 1 in the proof of Proposition 5.1 with the minor change. Similarly, we can apply the arguments in Step 4 and the first half of Step 5 in the proof of Proposition 5.1 with minor changes like above. Note that we can use Proposition 4.4 instead of Proposition 4.3 when we apply the arguments in Step 4 in the proof of Proposition 5.1. In this way, we can carry out the same discussions as in the proof of Proposition 5.1.

Therefore, by applying the proof of Proposition 5.1, we may prove that there is a good minimal model of (X, Δ) under the assumption that $D - aE$ and $D - (a - u)E$ are semi-ample, where $0 < a < e$ and $0 < u \ll a$ are rational numbers. Since $a < e$, $D - aE$ is big.

Step 2. Pick a sufficiently large and divisible positive integer m such that $a/(m+1) < u$ and $1/m < u$. Fix $A \sim_{\mathbb{Q}} m(D - aE)$ a general semi-ample \mathbb{Q} -divisor. Then A is big and we have

$$A + E \sim_{\mathbb{Q}} m(D - aE) + E = m\left(D - \left(a - \frac{1}{m}\right)E\right)$$

and

$$D + A \sim_{\mathbb{Q}} D + m(D - aE) = (m+1)\left(D - \left(a - \frac{a}{m+1}\right)E\right).$$

Since $0 < a/(m+1) < u$, $0 < 1/m < u$ and $D - aE$ and $D - (a - u)E$ are semi-ample, we see that $A + E$ and $D + A$ are semi-ample. By

Lemma 3.5, there is a sequence of birational maps of the D -MMP with scaling of A

$$(Z = Z_0, \lambda_0) \dashrightarrow \cdots \dashrightarrow (Z_i, \lambda_i) \dashrightarrow \cdots$$

such that $\lim_{i \rightarrow \infty} \lambda_i = 0$.

First we prove existence of a log minimal model of (X, Δ) . Take a dlt blow-up $f : (Y, \Gamma) \rightarrow (X, \Delta)$. Then we only have to prove that (Y, Γ) has a log minimal model. Set $g_0 = \pi \circ f : Y \rightarrow Z$ and $A' = g_0^* A$. By construction we have $K_Y + \Gamma \sim_{\mathbb{Q}} g_0^* D$. Since A is a general semi-ample divisor on Z , we may assume that $(Y, \Gamma + A')$ is also dlt. Set $G = f^* \Delta''$. By Lemma 3.6, we have the following diagram

$$\begin{array}{ccccccc} (Y = Y_0, \Gamma = \Gamma_0) & \rightarrow \cdots \rightarrow & (Y_{k_1}, \Gamma_{Y_{k_1}}) & \rightarrow \cdots \rightarrow & (Y_{k_i}, \Gamma_{Y_{k_i}}) & \rightarrow \cdots \\ g_0 \downarrow & & g_1 \downarrow & & g_i \downarrow & \\ (Z_0, \lambda_0) & \dashrightarrow \cdots \dashrightarrow & (Z_1, \lambda_1) & \dashrightarrow \cdots \dashrightarrow & (Z_i, \lambda_i) & \dashrightarrow \cdots \end{array}$$

such that

- (i) the upper horizontal sequence of birational maps is a sequence of the $(K_Y + \Gamma)$ -MMP with scaling of A' ,
- (ii) if we set $k_0 = 0$ and

$$\lambda'_j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{Y_j} + \Gamma_{Y_j} + \mu A'_{Y_j} \text{ is nef}\},$$

then $\lambda'_j = \lambda_i$ for any $i \geq 0$ and $k_i \leq j < k_{i+1}$, and

- (iii) $K_{Y_{k_i}} + \Gamma_{Y_{k_i}} \sim_{\mathbb{Q}} g_i^* D_{Z_i}$ and $G_{Y_{k_i}} \sim_{\mathbb{Q}} g_i^* E_{Z_i}$.

Step 3. In this step and the next step, we prove that the $(K_Y + \Gamma)$ -MMP with scaling of A' terminates.

Let C be any curve on Y_j contracted by the extremal contraction associated to $Y_j \dashrightarrow Y_{j+1}$. In this step we prove that $C \subset \text{Supp } G_{Y_j}$. If we can check this, we may prove that the above $(K_Y + \Gamma)$ -MMP occurs eventually disjoint from $\text{Supp } G$.

By the definition of the log MMP with scaling, $C \cdot (K_{Y_j} + \Gamma_{Y_j}) < 0$ and $C \cdot (K_{Y_j} + \Gamma_{Y_j} + \lambda'_j A'_{Y_j}) = 0$. Therefore $(C \cdot A'_{Y_j}) > 0$. Since $g_0^*(D - aE) \sim_{\mathbb{Q}} f^*(K_X + \Delta - a\Delta'') = K_Y + \Gamma - aG$ and by the definition of A , we have $A' \sim_{\mathbb{Q}} m(K_Y + \Gamma - aG)$. Then

$$a(C \cdot G_{Y_j}) = C \cdot (K_{Y_j} + \Gamma_{Y_j}) - \frac{1}{m}(C \cdot A'_{Y_j}) < 0.$$

Since $a > 0$ and G_{Y_j} is effective, we see that $C \subset \text{Supp } G_{Y_j}$.

Step 4. We apply the standard arguments of the special termination (cf. [F1]). Note that $\text{Supp } G \subset \text{Supp } \perp \Gamma \perp$. By replacing (Y, Γ) with

$(Y_{k_i}, \Gamma_{Y_{k_i}})$ for some $i \gg 0$, we may assume that the $(K_Y + \Gamma)$ -MMP contains only flips and flipping locus on each flip contains no lc centers.

Pick a positive integer r and suppose that the $(K_Y + \Gamma)$ -MMP occurs eventually disjoint from any lc center of (Y, Γ) in $\text{Supp } G$ whose dimension is less than r . Let $S \subset \text{Supp } G$ be an r -dimensional lc center of (Y, Γ) and S_j be the birational transform of S on Y_j for any j . We define \mathbb{Q} -divisors Γ_{S_j} by the adjunction $(K_{Y_j} + \Gamma_{Y_j})|_{S_j} = K_{S_j} + \Gamma_{S_j}$. Then (S_j, Γ_{S_j}) is dlt, and we see that $\phi_j : S_j \dashrightarrow S_{j+1}$ is a birational contraction and $\phi_{j*}(K_{S_j} + \Gamma_{S_j}) = K_{S_{j+1}} + \Gamma_{S_{j+1}}$ for any $j \gg 0$ (cf. [F1]). By replacing (Y, Γ) with (Y_{k_i}, Γ_{k_i}) for some $i \gg 0$, we may assume that S_j satisfies the above properties for any $j \geq 0$ and the $(K_Y + \Gamma)$ -MMP occurs disjoint from any lc center of (Y, Γ) in $\text{Supp } G$ whose dimension is less than r . Let $(T, \Theta) \rightarrow (S, \Gamma_S)$ be a dlt blow-up and A'' be the pullback of A' . Since A' is semi-ample, by replacing A'' if necessary, we may assume that A'' is effective and $(T, \Theta + A'')$ is dlt. Set $T_0^0 = T$ and $\Theta_{T_0^0} = \Theta$. By the same arguments as in the proof of Lemma 3.6 (see also [F1]), we get the following diagram

$$\begin{array}{ccccccc} (T_0^0, \Theta_{T_0^0}) & \dashrightarrow & \cdots & \dashrightarrow & (T_i^j, \Theta_{T_i^j}) & \dashrightarrow & \cdots & \dashrightarrow & (T_i^{l_i} = T_{i+1}^0, \Theta_{T_{i+1}^0}) & \dashrightarrow & \cdots \\ \downarrow & & & & & & & & \downarrow & & \\ (Y, \Gamma, \lambda_0') & \dashrightarrow & \cdots & \dashrightarrow & & \dashrightarrow & & & (Y_{i+1}, \Gamma_{Y_{i+1}}, \lambda_{i+1}') & \dashrightarrow & \cdots \end{array}$$

such that

- (i) the upper horizontal sequence of birational maps is a sequence of steps of the $(K_T + \Theta)$ -MMP with scaling A'' , and
- (ii) the morphism $T_i^0 \rightarrow Y_i$ is the composition of a dlt blow-up of (S, Γ_{S_i}) and the inclusion $S_i \hookrightarrow Y_i$.

By construction, we also have the following property.

- (iii) If we set

$$\lambda_i^j = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{T_i^j} + \Theta_{T_i^j} + \mu A''_{T_i^j} \text{ is nef, } 0 \leq j < l_i\},$$

$$\text{then } \lambda_i^0 \leq \lambda_i'.$$

Note that we may have $\lambda_i^0 < \lambda_i'$ because the morphism $T_i^0 \rightarrow Y_i$ is not surjective. If $\lambda_i^0 < \lambda_i'$, then we have $T_i^0 \simeq T_i^1 \simeq \cdots \simeq T_i^{l_i} = T_{i+1}^0$ by construction.

By (iii) of the above properties, $K_{T_i^0} + \Theta_{T_i^0} + \lambda_i^0 A''_{T_i^0}$ is pseudo-effective. Then $K_T + \Theta + \lambda_i^0 A''$ is also pseudo-effective. Since $\lim_{i \rightarrow \infty} \lambda_i^0 = 0$, we see that $K_T + \Theta$ is pseudo-effective. Now consider the composition of morphisms $T \rightarrow S \hookrightarrow Y \rightarrow Z$, which we denote $h : T \rightarrow Z$. Recall that $S \subset \text{Supp } G$ and that $g_0(\text{Supp } G) \subsetneq Z$. Let Z_T be the normalization

of $h(T)$. Then $K_T + \Theta \sim_{\mathbb{Q}, Z_T} 0$ because $K_Y + \Gamma_Y \sim_{\mathbb{Q}, Z} 0$. Moreover $\dim Z_T < \dim Z$ by construction. Since we assume Theorem 4.2 for $d_0 - 1$, applying the hypothesis to $(T, \Theta) \rightarrow Z_T$, we see that (T, Θ) has a good minimal model. Then the above $(K_T + \Theta)$ -MMP with scaling terminates by [B2, Theorem 4.1 (iii)] because $\lim_{i \rightarrow \infty} \lambda'_i = 0$. By the same arguments as in [F1], the $(K_Y + \Gamma)$ -MMP with scaling occurs eventually disjoint from S . Therefore, by the induction on r , we see that the $(K_Y + \Gamma)$ -MMP occurs eventually disjoint from $\text{Supp } G$. In this way we see that the $(K_Y + \Gamma)$ -MMP with scaling of A' constructed in Step 2 must terminate.

Step 5. Finally we prove that (X, Δ) has a good minimal model. By Step 4, the D -MMP with scaling of A constructed in Step 2 terminates (cf. Lemma 3.6). By Lemma 3.6, we have the following diagram

$$\begin{array}{ccc} (X, \Delta) & \dashrightarrow \cdots \dashrightarrow & (X_{k_i}, \Delta_{X_{k_i}}) \\ \pi \downarrow & & \downarrow \pi_i \\ (Z, \lambda_0) & \dashrightarrow \cdots \dashrightarrow & (Z_i, \lambda_i = 0) \end{array}$$

such that $(X_{k_i}, \Delta_{X_{k_i}})$ is a log minimal model of (X, Δ) . We note that $K_{X_{k_i}} + \Delta_{X_{k_i}} \sim_{\mathbb{Q}} \pi_i^* D_{Z_i}$ and D_{Z_i} is big. Then $K_{X_{k_i}} + \Delta_{X_{k_i}}$ is semi-ample by Lemma 5.5 below. So we are done. \square

Lemma 5.5. *Fix a positive integer d_0 . Assume Theorem 4.2 for $d_0 - 1$, and assume existence of a good minimal model or a Mori fiber space for all projective Kawamata log terminal pairs of dimension d with boundary \mathbb{Q} -divisors such that $d \leq d_0 - 1$.*

Let $\pi : X \rightarrow Z$ be a projective surjective morphism of a normal projective varieties such that $\dim Z \leq d_0$, and (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor. Suppose that $K_X + \Delta$ is nef and $K_X + \Delta \sim_{\mathbb{Q}} \pi^ D$ for a big \mathbb{Q} -Cartier \mathbb{Q} -divisor D on Z .*

Then $K_X + \Delta$ is semi-ample.

Proof. Suppose that there is an lc center S of (X, Δ) dominating Z . By taking an appropriate dlt blow-up of (X, Δ) and replacing (X, Δ) with the dlt model, we may assume that S is a component of $\lfloor \Delta \rfloor$ and $(S, \text{Diff}(\Delta - S))$ is dlt. Then $K_S + \text{Diff}(\Delta - S) \sim_{\mathbb{Q}} (\pi|_S)^* D$ and $K_X + \Delta$ is semi-ample if and only if $K_S + \text{Diff}(\Delta - S)$ is semi-ample. Therefore we may replace $\pi : (X, \Delta) \rightarrow Z$ with $\pi|_S : (S, \text{Diff}(\Delta - S)) \rightarrow Z$. By repeating the discussion, we can assume that there is no lc centers of (X, Δ) dominating Z . By taking a dlt blow-up, we may assume that

X is \mathbb{Q} -factorial and $(X, 0)$ is klt. Moreover, by the same arguments as in the proof of Lemma 4.5, we can assume that

- (i) π has connected fibers and Z is \mathbb{Q} -factorial,
- (ii) $(X, \Delta = \Delta' + \Delta'')$ satisfies all the conditions of Lemma 2.13 with respect to π , and
- (iii) $\Delta'' \sim_{\mathbb{Q}} \pi^* E$ for some effective \mathbb{Q} -divisor on Z .

Note that the bigness of D still holds after this process and (X, Δ') is klt by construction. Pick a rational number $\epsilon > 0$ sufficiently small satisfying that $D - \epsilon E$ is big. By the same arguments as in the Step 1 in the proof of Proposition 5.1 (see also Step 1 in the proof of Proposition 5.4), we have the following diagram

$$\begin{array}{ccc} (X, \Delta - \epsilon \Delta'') & \dashrightarrow & (X', \Delta_{X'} - \epsilon \Delta''_{X'}) \\ \pi \downarrow & & \pi' \downarrow \\ Z & \dashrightarrow & Z' \end{array}$$

such that $(X', \Delta_{X'} - \epsilon \Delta''_{X'})$ is a good minimal model of $(X, \Delta - \epsilon \Delta'')$. Since we pick $\epsilon > 0$ sufficiently small, by the standard argument of the length of extremal rays (cf. the proof of [H, Theorem 1.2]), $K_{X'} + \Delta_{X'}$ is nef and $K_X + \Delta$ is semi-ample if and only if $K_{X'} + \Delta_{X'}$ is semi-ample. We can check that $(X', \Delta_{X'} - \epsilon \Delta''_{X'})$ is klt and $\pi'(\text{Supp } \Delta''_{X'}) \subsetneq Z'$. If $\Delta''_{X'} = 0$, then $K_{X'} + \Delta_{X'} = K_{X'} + \Delta'_{X'} - \epsilon \Delta''_{X'}$ is semi-ample. Therefore we may assume that $\Delta''_{X'} \neq 0$.

Let $f : (Y, \Gamma) \rightarrow (X', \Delta_{X'})$ be a dlt blow-up and we set $G = f^* \Delta''_{X'}$. Since $(X', \Delta_{X'} - \epsilon \Delta''_{X'})$ is klt and $\Delta''_{X'}$ is a reduced divisor, we see that $\text{Supp } G = \text{Supp } \Gamma$. Moreover $K_Y + \Gamma - \epsilon' G$ is semi-ample for any sufficiently small rational number $\epsilon' > 0$ because $K_{X'} + \Delta_{X'} - \epsilon' \Delta''_{X'}$ is semi-ample by Proposition 4.4. Let T be a component of G , and we define the \mathbb{Q} -divisor Θ by the adjunction $(K_Y + \Gamma)|_T = K_T + \Theta$. By construction, (T, Θ) is dlt and $(\pi' \circ f)(T) \subsetneq Z'$. Let Z_T be the normalization of $(\pi' \circ f)(T)$. Then $\dim Z_T < \dim Z' \leq d_0$. Therefore, by applying Theorem 4.2 for $d_0 - 1$ to the morphism $(T, \Theta) \rightarrow Z_T$, we see that $K_T + \Theta = (K_Y + \Gamma)|_T$ is semi-ample. Since T is any component of G and $K_Y + \Gamma - \epsilon' G$ is semi-ample for any sufficiently small rational number $\epsilon' > 0$, by [FG2, Theorem 1.4] and the arguments of [KMM, Section 7] (cf. [B2, Theorem 1.7]), we see that $K_Y + \Gamma$ is semi-ample. Then $K_{X'} + \Delta_{X'}$ is semi-ample and thus $K_X + \Delta$ is semi-ample. So we are done. \square

6. PROOF OF THE COROLLARY

In this section we prove Corollary 1.4.

Theorem 6.1. *Let $\pi : X \rightarrow Z$ be a projective surjective morphism of normal projective varieties and let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor. Suppose that $K_X + \Delta \sim_{\mathbb{Q}} \pi^* D$ for a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on Z .*

If $\dim Z \leq 3$ or $\dim Z = 4$ and D is big, then (X, Δ) has a good minimal model or a Mori fiber space.

Proof. By the same arguments as in the proof of Lemma 4.5, we can assume that $\pi : (X, \Delta) \rightarrow Z$ satisfies all the conditions of Lemma 4.5. Note that bigness of D still holds after the process. Since the log MMP and the abundance conjecture hold for all log canonical threefolds, the theorem follows from Theorem 4.2 and Proposition 5.4. \square

Theorem 6.2. *Let (X, Δ) be a projective log canonical pair such that Δ is a \mathbb{Q} -divisor and the log Kodaira dimension $\kappa(X, K_X + \Delta)$ is non-negative. Let F be the general fiber of the Iitaka fibration and (F, Δ_F) be the restriction of (X, Δ) to F . Suppose that (F, Δ_F) has a good minimal model.*

If (X, Δ) is Kawamata log terminal or $\kappa(X, K_X + \Delta) \leq 4$, then (X, Δ) has a good minimal model.

Proof. Let $f : X \dashrightarrow W$ be the Iitaka fibration. Taking an appropriate resolution of X if necessary, we may in particular assume that X is \mathbb{Q} -factorial, $(X, 0)$ is klt and f is a morphism. By [K, Theorem 2] (see also [AK, Theorem 0.3]), we can in particular assume that W is smooth and all fibers have the same dimension (cf. [HX, Theorem 2.1]). By construction, there is an effective \mathbb{Q} -divisor E such that $K_X + \Delta \sim_{\mathbb{Q}} E$. Then we can write $E = E^h + E^v$, where every component of E^h dominates W and E^v is vertical. Since all fibers of f have the same dimension, the image of any component of E^v on W is a divisor. Since W is smooth, we can consider

$$\mu_B = \sup\{\mu \mid E^v - \mu f^* B \text{ is effective}\}$$

for any prime divisor B on W . Then it is easy to see that μ_B is a rational number for any B and there are only finitely many divisors B such that $\mu_B > 0$. Set

$$B' = \sum_B \mu_B B \quad \text{and} \quad E' = E^v - f^* B'.$$

Then $K_X + \Delta \sim_{\mathbb{Q}, W} E^h + E'$ and E' is effective. Moreover we see that E' is very exceptional over W (cf. [B2, Definition 3.1]).

We run the $(K_X + \Delta)$ -MMP over W with scaling of an ample divisor

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots$$

Let $f_i : X_i \rightarrow W$ be the induced morphism and let F_i be the general fiber of f_i . Recall that (F, Δ_F) has a good minimal model by the hypothesis. Since $\kappa(F_i, (K_{X_i} + \Delta_{X_i})|_{F_i}) = 0$, we have

$$(K_{X_i} + \Delta_{X_i})|_{F_i} \sim_{\mathbb{Q}} E_{X_i}|_{F_i} \sim_{\mathbb{Q}} (E_{X_i}^h + E'_{X_i})|_{F_i} \sim_{\mathbb{Q}} 0$$

for any $i \gg 0$. Therefore $E_{X_i}^h + E'_{X_i}$ is vertical and thus we have $K_{X_i} + \Delta_{X_i} \sim_{\mathbb{Q}, W} E'_{X_i}$. We note that E'_{X_i} is very exceptional over W because the $(K_X + \Delta)$ -MMP occurs only in $\text{Supp}(E^h + E')$. Moreover $K_{X_i} + \Delta_{X_i} \sim_{\mathbb{Q}, W} E'_{X_i}$ is the limit of movable divisors over W for any $i \gg 0$. Then $E'_{X_i} = 0$ by [B2, Lemma 3.3]. Therefore $K_{X_i} + \Delta_{X_i} \sim_{\mathbb{Q}, W} 0$ for some i . Let D be a \mathbb{Q} -divisor on W such that $K_{X_i} + \Delta_{X_i} \sim_{\mathbb{Q}} f_i^* D$. Then D is big since $\kappa(W, D) = \kappa(X_i, K_{X_i} + \Delta_{X_i}) = \dim W$.

If (X, Δ) is klt, then (X_i, Δ_{X_i}) is also klt and (X_i, Δ_{X_i}) has a good minimal model by Proposition 4.4. Note that W is in particular \mathbb{Q} -factorial from our assumption. On the other hand, if $\kappa(X, K_X + \Delta) \leq 4$, then $\dim W \leq 4$ and therefore (X_i, Δ_{X_i}) has a good minimal model by Theorem 6.1.

Therefore we see that (X, Δ) has a good minimal model. \square

Proof of Corollary 1.4. Since (X, Δ) is not of log general type, we have $\kappa(X, K_X + \Delta) \leq 4$. We can assume that $\kappa(X, K_X + \Delta) > 0$ because otherwise the statement is obvious.

Suppose that $\kappa(X, K_X + \Delta) = 1$. Then we take the Iitaka fibration $f : X \dashrightarrow W$. By taking an appropriate resolution of X if necessary, we may assume that f is a morphism. In general, there is a \mathbb{Q} -divisor B on W and a positive integer m such that $\mathcal{R}(X, m(K_X + \Delta)) \simeq \mathcal{R}(W, mB)$ (cf. [FM]). Since W is a smooth curve, $\mathcal{R}(W, mB)$ is finitely generated. Then $\mathcal{R}(X, m(K_X + \Delta))$ is finitely generated, and thus $\mathcal{R}(X, K_X + \Delta)$ is finitely generated.

Suppose that $\kappa(X, K_X + \Delta) \geq 2$. Let F be the general fiber of the Iitaka fibration and (F, Δ_F) be the restriction of (X, Δ) . Then (F, Δ_F) is lc and $\dim F \leq 3$ by construction, and thus (F, Δ_F) has a good minimal model. By Theorem 6.2, we see that (X, Δ) has a good minimal model. Thus the log canonical ring $\mathcal{R}(X, K_X + \Delta)$ is finitely generated. \square

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: `hkenta@math.kyoto-u.ac.jp`